

THE CUBIC HECKE ALGEBRA ON AT MOST 5 STRANDS

IVAN MARIN

To the memory of Johann Gustav Hermes, who worked 10 years on completing the construction of the 65537-gon and on producing the corresponding beautiful artwork of drawings and numbers, these days known as ‘Der Koffer’ in Göttingen’s library.

ABSTRACT. We prove that the quotient of the group algebra of the braid group on 5 strands by a generic cubic relation has finite rank. This was conjectured in 1998 by Broué, Malle and Rouquier and has for consequence that this algebra is a flat deformation of the group algebra of the complex reflection group G_{32} , of order 155,520.

1. INTRODUCTION

In 1957 H.S.M. Coxeter proved (see [7]) that the quotient of the braid group B_n on $n \geq 2$ strands by the relations $s_i^k = 1$, where s_1, \dots, s_{n-1} denote the usual Artin generators, is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. This means that, besides the obvious case $k = 2$, which leads to the symmetric group, and the case $n = 2$, there is only a finite number of such groups. They all turn out to be irreducible complex reflection groups, namely finite subgroups of $\mathrm{GL}_n(\mathbf{C})$ generated by endomorphisms which fix an hyperplane (so-called pseudo-reflections), and which leave no proper subspace invariant. In the classical classification of such objects, due to Shephard and Todd, they are nicknamed as G_4, G_8, G_{16} for $n = 3$ and $k = 3, 4, 5$, G_{25}, G_{32} for $n = 4, 5$ and $k = 3$.

In 1998, M. Broué, G. Malle and R. Rouquier conjectured (see [4]) that the group algebra of complex reflection groups admit flat deformations similar to the Hecke algebra of a Weyl or Coxeter group. They actually introduced natural deformations of such group algebras, called them the (generic) Hecke algebra associated to such a group, and they conjectured that these were flat deformations, and in particular that they have finite rank. For the groups we are interested in, this conjecture actually amounts to saying that the quotients of the group algebra RB_n by the relations $s_i^k + a_{k-1}s_i^{k-1} + \dots + a_1s_i + a_0 = 0$, where $R = \mathbf{Z}[a_{k-1}, \dots, a_1, a_0, a_0^{-1}]$, is a flat deformation of the group algebra RW , where $W = B_n/s_i^k$ (note that we actually use a slightly smaller ring than the one used in [4] and [3]). This conjecture was proved in [3] for all the five groups above but the largest case G_{32} (the proof for G_{25} is however only sketched there).

According to [4] (see the proof of theorem 4.24 there) only the following needs to be proved : that the algebra is spanned over R by $|W|$ elements. This is what we prove here.

Theorem 1.1. *The generic Hecke algebra associated to $W = G_{32}$ is spanned by $|W|$ elements, and is thus a free R -module of rank $|W|$ which becomes isomorphic to the group algebra of W after a suitable extension of scalars.*

More precisely, according to [10] corollary 7.2, a convenient extension of scalars would be $\mathbf{Q}(\zeta_3, (\zeta_3^{-r}u_r)^{\frac{1}{6}}, r = 0, 1, 2)$ where ζ_3 is a primitive 3rd root of 1 and $X^3 + a_2X^2 + a_1X + a_0 = (X - u_0)(X - u_1)(X - u_2)$ or, better, the algebraic extension of $\mathbf{Q}(\zeta_3)(u_0, u_1, u_2)$ generated by $\sqrt{u_0u_1}$ and $\sqrt[3]{u_0u_1u_2}$ (see [10] table 8.2 and proposition 5.1).

In the general setting of complex reflection groups, it is known that this conjecture is true

- for the general series (usually denoted $G(de, e, r)$) of complex reflection groups (by work of Ariki and Ariki-Koike),
- for most of the exceptional groups of rank 2 by [3] and [12], which are numbered G_4 to G_{22} , and by [8] for all exceptional groups of rank 2 over a larger ring than expected,

- for the Coxeter groups.

The remaining cases are in rank 4 the groups G_{29} ([12] however proves it over the field of fraction by computer means), G_{31} , G_{32} , in rank 5 the group G_{33} and in rank 6 the group G_{34} . All but G_{32} , whose case we settled here, have all their pseudo-reflections of order 2.

In the case studied here, we actually prove more. Here and in the sequel we denote A_n the quotient of RB_n by the generic cubic relation $s_i^3 - as_i^2 - bs_i - c = 0$. The usual embedding $B_n \hookrightarrow B_{n+1}$ induces a natural morphism $A_n \rightarrow A_{n+1}$, hence a A_n -bimodule structure on A_{n+1} . For $n \leq 4$, we give a decomposition of A_{n+1} as A_n -bimodule. This immediately provides an explicit R -basis of A_n for $n \leq 5$, made of images of braids in B_n . Recall that the orders of G_4 , G_{25} and G_{32} are 24, 648, 155520.

The following theorem is a recollection of the main results of this article : see in particular theorems 3.2, 4.1, 6.21 and 6.26 as well as corollary 5.12, and recall that the argument of [4] theorem 4.24 (which involves a transcendental monodromy construction) shows that proving that the Hecke algebra of type W is R -generated by $|W|$ elements ensures that this Hecke algebra is free as a R -module, with basis the given $|W|$ elements. Moreover, notice that, if we have an inclusion of parabolic subgroups $W_0 \subset W$ with corresponding Hecke algebras $H_0 \subset H$, knowing the conjecture for H_0 and that H is generated by $|W/W_0|$ elements as a H_0 -module proves (1) the conjecture for H and (2) that H is free as a H_0 -module, with basis these elements. Indeed, letting $N = |W/W_0|$ the assumption provides a H_0 -module morphism $H_0^N \rightarrow H$; composing with $(R^{|W_0|})^N \simeq H_0^N$ this yields a surjective morphism $R^{|W|} \rightarrow H$ which is an isomorphism by the argument of [4]. This proves that the original morphism $H_0^N \rightarrow H$ has no kernel either, and so is an isomorphism.

Theorem 1.2.

- Let $S_2 = \{1, s_1, s_1^{-1}\} \subset B_2$. One has $|S_2| = 3$ and S_2 provides an R -basis of A_2 .
- Let $S_3 = S_2 \sqcup S_2 s_2^\pm S_2 \sqcup S_2 s_2^{-1} s_1 s_2^{-1} \subset B_3$. One has $|S_3| = 24$ and S_3 provides a R -basis of A_3 .
- A_4 is a free A_3 -module of rank 27. A basis of this A_3 -module is provided by elements of the braid group (including 1) which map to a system of representatives of G_{25}/G_4 .
- A_4 is a free R -module of rank 648. A basis of this R -module is provided by elements of the braid group including 1 which map to all G_{25} .
- A_4 is a free $A_2 \otimes_R A_2 \simeq \langle s_1, s_3 \rangle$ -module of rank 72. A basis of this $\langle s_1, s_3 \rangle$ -module is provided by elements of the braid group including 1 which map to a system of representatives of $G_{25}/(\mathbf{Z}/3\mathbf{Z})^2$.
- A_5 is a free A_4 -module of rank 240. A basis is provided by elements of the braid group including 1 which map to a system of representatives of G_{32}/G_{25} .
- A_5 is a free R -module of rank 155,520. A basis of this R -module is provided by elements of the braid group which include 1 and which map to all G_{32} .

Corollary 1.3. *The natural map $A_n \rightarrow A_{n+1}$ is injective for $2 \leq n \leq 4$.*

We describe the plan of the proof. Our method is inductive. We find generators of A_{n+1} as a A_n -bimodule, and only then as a A_n -module. After some preliminaries in section 2 we do the case of A_3 in section 3. The structure of A_4 as a A_3 -module is obtained in section 4. Before considering A_5 , we provide in section 5 an alternative description of A_4 , this time as a $\langle s_1, s_3 \rangle$ -module. In addition to providing an alternative proof of the conjecture for A_4 , this is used in the decomposition of A_5 as a A_4 -module. This decomposition is obtained in section 6. We first obtain a decomposition of A_5 as a A_4 -bimodule, and introduce a filtration of A_5 by simpler A_4 -bimodules. The latest step of the filtration has original generators originating from the center of the braid group, and this turns out to be the crucial reason why this filtration terminates, thus proving that A_5 is a R -module of finite rank. For proving this crucial property one needs a lengthy calculation which is postponed in section 7. We conclude the section 6 and the proof of the main theorem by studying the structure as A_4 -modules of the A_4 -bimodules involved there.

1.1. Perspectives. It seems likely that our methods can be used to attack the conjecture for other complex reflection groups of higher rank. One indeed has the following standard inclusions

of parabolic subgroups (except for the dotted line, which is not a parabolic inclusion). The number associated to the inclusion is the number of double classes. Note again that the groups of rank at least 3 for which the conjecture remains open have all their reflections of order 2.

$$\begin{array}{ccccccc}
 G(3, 3, 2) & \xrightarrow{4} & G(3, 3, 3) & \xrightarrow{4} & G(3, 3, 4) & \xrightarrow{6} & G_{33} \xrightarrow{13} G_{34} \\
 & & & & & & \\
 G(4, 4, 2) & \xrightarrow{5} & G(4, 4, 3) & \xrightarrow{9} & G_{29} & & \\
 & & & \nearrow 16 & \vdots 2 & & \\
 & & G(2, 1, 3) & & G_{31} & &
 \end{array}$$

For instance, 8 of the 9 double classes of $W = G_{29} = \langle g_1, g_2, g_3, g_4 \rangle$ with respect to $W_0 = G(4, 4, 3) = \langle g_2, g_3, g_4 \rangle$ have for representatives $g_1^\varepsilon z$ for $z \in Z(W)$ and $\varepsilon \in \{0, 1\}$. If we had a practical knowledge of the braid groups of type G_{29} and $G(4, 4, 3)$ of the same level than the one we have for the usual braid group, the methods used here would then probably yield a proof of the conjecture for G_{29} in the same way we managed to get one for G_{32} , as this kind of phenomenon (that the most complicated double classes are mainly represented by central elements) is crucial in our proof. Similarly, if $G_{34} = \langle s_1, \dots, s_6 \rangle$ with $G_{33} = \langle s_1, \dots, s_5 \rangle$, one can check that 12 of the 13 double classes have for representative a term of the form zs_6^ε for $\varepsilon \in \{0, 1\}$ and z a central element of G_{34} .

Another natural question is whether similar deformations exist for a higher number of strands. Indeed, although it is known that the groups $\Gamma_n = B_n/s_i^3$ are infinite for $n \geq 6$, it was proved in [1] (see also [5]) that $\Gamma_n^{(1)} = \Gamma_n/z_5^2$ and $\Gamma_n^{(2)} = \Gamma_n/z_5^3$ are finite for arbitrary $n \geq 5$, and are related to symplectic group over \mathbf{F}_3 and to unitary groups over \mathbf{F}_2 , respectively. Here z_5 denotes the image of the generator $(s_1 s_2 s_3 s_4)^5$ of the braid group on 5 strands into $\Gamma_n, n \geq 5$, which has order 6 in Γ_5 . It is thus tempting to look for deformations of the group algebras of $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$ for arbitrary n that would be quotients of the group algebra of the braid group by a generic cubic relations *and* other relations probably involving z_5 .

1.2. Applications. We mention the following consequences. A first one concerns the study of linear representations of the (usual) braid groups. A consequence of the proof in [3] for the cases G_4, G_8 and G_{16} was a classification of the linear representations of the braid group B_3 in which the image of s_1 (and thus of all s_i) is killed by a polynomial of degree at most 5 : indeed, such a representation has to factorize through the corresponding Hecke algebra. This proves that such representations have a very rigid structure, a result rediscovered in [13]. A similar consequence of this new result is a classification of the linear representations of the braid group B_n for n at most 5 in which the image of s_1 is killed by a cubic polynomial.

A second one is about the cubic invariants of knots and links. The algebras connected to cubic invariants, including the Kauffman polynomial and the Links-Gould polynomial, are quotients of A_n . Our result gives the structure of A_5 ; in order to prove it, we actually establish its decomposition as a A_4 -bimodule, which may be useful in order to understand the possible Markov traces factorizing through A_n .

Specifically, in [5], we used the representation theory of the group G_{32} to prove that an algebra $K_n(\gamma)$ introduced by L. Funar in [9] for studying knot invariants collapsed for large n over a field of characteristic distinct from 2, and in characteristic 0 for $n \geq 5$. An immediate consequence of the present result is that our argument in characteristic 0 applies verbatim to prove that the deformation $K_n(\alpha, \beta)$ introduced by P. Bellingeri and L. Funar in [2] also collapses for $n \geq 5$. We provide the details below.

Theorem 1.4. *The generic algebra $K_n(\alpha, \beta)$ introduced in [2] is zero for $n \geq 5$.*

Proof. We use the notations of [2]. Let \mathbf{k} be an algebraically closed extension of $\mathbf{Q}(\alpha, \beta)$. The $\mathbf{Z}[\alpha, \beta]$ -algebra $K_n(\alpha, \beta)$ is defined as the quotient of the group algebra $\mathbf{Z}[\alpha, \beta]B_n$ by the two-sided ideal generated by the elements $s_i^3 - \alpha s_i^2 - \beta s_i - 1$ and another element $q \in \mathbf{Z}[\alpha, \beta]B_3 \subset \mathbf{Z}[\alpha, \beta]B_n$.

We let $\varphi : \mathbf{Z}[a, b, c, c^{-1}] \rightarrow \mathbf{Z}[\alpha, \beta]$ be the specialization $a \mapsto \alpha$, $b \mapsto \beta$, $c \mapsto 1$, and let A_n^0 denote $A_n \otimes_{\varphi} \mathbf{Z}[\alpha, \beta]$. Obviously $K_n(\alpha, \beta)$ is a quotient of A_n^0 , more precisely the quotient of A_n^0 by the two-sided ideal generated by (the canonical image of) q . Let \mathbf{k} denote an algebraically closed extension of $\mathbf{Q}(\alpha, \beta)$. We have $A_3^0 \otimes_{\mathbf{Z}[\alpha, \beta]} \mathbf{k} \simeq \mathbf{k}G_4 \simeq \mathbf{k}^3 \oplus \text{Mat}_2(\mathbf{k})^3 \oplus \text{Mat}_3(\mathbf{k})$, and the ideal generated by q is by definition the factor \mathbf{k}^3 in this decomposition (see remark 1.3 in [2]). As a consequence, the \mathbf{k} -algebra $\mathbf{k}K_5(\alpha, \beta)$ is the quotient of the semisimple algebra $\mathbf{k}A_5^0 \simeq \mathbf{k}G_{32}$ by the following two-sided ideal : make the direct sum of all the direct factors $\text{Mat}_N(\mathbf{k})$ whose corresponding irreducible representations have at least one 1-dimensional component in their restriction to $\mathbf{k}A_3^0$. Now, to the expense of possibly enlarging \mathbf{k} , the isomorphisms between the algebras A_n^0 and the corresponding group algebras can be chosen in such a way that the following diagram commutes (e.g. by theorem 2.9 of [11] – see also remark 2.11 there).

$$\begin{array}{ccccc} \mathbf{k}A_3^0 & \longrightarrow & \mathbf{k}A_4^0 & \longrightarrow & \mathbf{k}A_5^0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k}G_4 & \longrightarrow & \mathbf{k}G_{25} & \longrightarrow & \mathbf{k}G_{32} \end{array}$$

As in [5], the induction table between the (ordinary) characters of G_4 of G_{32} then shows that *all* direct factors $\text{Mat}_N(\mathbf{k})$ satisfy this property, and thus the two-sided ideal is all A_5^0 . It follows that $K_5(\alpha, \beta) = 0$, whence $K_n(\alpha, \beta) = 0$ for $n \geq 5$, as $K_n(\alpha, \beta)$ is generated by conjugates of the image of $K_5(\alpha, \beta)$. □

2. PRELIMINARIES AND NOTATIONS

We let $R = \mathbf{Z}[a, b, c, c^{-1}]$ and let B_n denote the braid group on n strands, generated by the braids s_1, \dots, s_{n-1} with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|j - i| \geq 2$. The cubic Hecke algebra A_n for $n \geq 2$ is the quotient of the group algebra RB_n by the relations $s_i^3 = as_i^2 + bs_i + c$. We identify s_i to their images in A_n . Notice that, since c is invertible in R , s_i is still invertible, and we have the equivalent relations $s_i^2 = as_i + b + cs_i^{-1}$, etc.

The group algebra RB_n admits the automorphism $s_i \mapsto s_{n-i}$, which induces an automorphism of A_n , as a R -algebra. The automorphism $s_i \mapsto s_i^{-1}$ of B_n induces an automorphism Φ of A_n as a \mathbf{Z} -algebra, defined by $s_i \mapsto s_i^{-1}$, $a \mapsto -bc^{-1}$, $b \mapsto -ac^{-1}$, $c \mapsto c^{-1}$, and similarly the skew-automorphism Ψ of B_n defined by $s_i \mapsto s_i^{-1}$ induces a skew-automorphism of A_n as a \mathbf{Z} -algebra.

In the sequel we will denote u_i the R -subalgebra of A_n generated by s_i (or equivalently by s_i^{-1}).

The following equalities hold in the braid group, and thus also in A_n . We state them as a lemma because of their importance in the sequel. Notice that they transform an element of the form $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp}$ into an element of $u_i u_{i+1} u_i$.

Lemma 2.1. *For $\alpha \in \{-1, 1\}$, we have $s_{i+1}^{\alpha} s_i^{\epsilon} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{\alpha} s_i^{\alpha}$ and $s_{i+1}^{\alpha} s_i^{-\alpha} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{-\alpha} s_i^{\alpha}$, that is*

$$\begin{aligned} s_{i+1} s_i s_{i+1}^{-1} &= s_i^{-1} s_{i+1} s_i \\ s_{i+1} s_i^{-1} s_{i+1}^{-1} &= s_i^{-1} s_{i+1}^{-1} s_i \\ s_{i+1}^{-1} s_i s_{i+1} &= s_i s_{i+1} s_i^{-1} \\ s_{i+1}^{-1} s_i^{-1} s_{i+1} &= s_i s_{i+1}^{-1} s_i^{-1} \end{aligned}$$

Lemma 2.2.

- (1) $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp} \in u_i u_{i+1} u_i$
- (2) $s_{i+1}^{\pm} s_i^{\pm} s_{i+1}^{\epsilon} \in u_i u_{i+1} u_i$
- (3) $s_{i+1}^{\epsilon} s_i^{\pm} s_{i+1}^{\pm} \in u_i u_{i+1} u_i$

Proof. The first item is a direct consequence of lemma 2.1, and the latter two items are consequences of (1) and of the braid relations $s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm} = s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm}$. □

Lemma 2.3.

- (1) For all $x \in u_i$, $(s_{i+1}^{-1}s_i s_{i+1}^{-1})x \in x(s_{i+1}^{-1}s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$.
 (2) For all $x \in u_i$, $(s_{i+1}s_i^{-1}s_{i+1})x \in x(s_{i+1}s_i^{-1}s_{i+1}) + u_i u_{i+1} u_i$.

Proof. (2) is a consequence of (1) up to applying an automorphism of A_n , so we restrict ourselves to proving (1). Since u_i is generated as a R -algebra by s_i^{-1} , we only need to prove $(s_{i+1}^{-1}s_i s_{i+1}^{-1})s_i^{-1} \in s_i^{-1}(s_{i+1}^{-1}s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$. We use $s_i = cs_i^{-2} + bs_i^{-1} + a$, $s_i^{-2} = c^{-1}s_i - ac^{-1} - bc^{-1}s_i^{-1}$ and the braid relations, and get

$$\begin{aligned}
 (s_{i+1}^{-1}s_i s_{i+1}^{-1})s_i^{-1} &= s_{i+1}^{-1}s_i s_{i+1}^{-1}s_i^{-1} \\
 &= s_{i+1}^{-1}(cs_i^{-2} + bs_i^{-1} + a)s_{i+1}^{-1}s_i^{-1} \\
 &= cs_{i+1}^{-1}s_i^{-2}s_{i+1}^{-1}s_i^{-1} + bs_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}s_i^{-1} + as_{i+1}^{-1}s_i^{-1}s_i^{-1} \\
 &= cs_{i+1}^{-1}s_i^{-2}s_{i+1}^{-1}s_i^{-1} + bs_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}s_i^{-1} + as_{i+1}^{-1}s_i^{-1}s_i^{-1} \\
 &\in cs_{i+1}^{-1}s_i^{-1}(s_{i+1}^{-1}s_i^{-1}s_i^{-1}) + u_i u_{i+1} u_i \\
 &\in c(s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1})s_i^{-1}s_i^{-1} + u_i u_{i+1} u_i \\
 &\in cs_{i+1}^{-1}s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1} + u_i u_{i+1} u_i \\
 &\in cs_{i+1}^{-1}s_{i+1}^{-1}(c^{-1}s_i - ac^{-1} - bc^{-1}s_i^{-1})s_{i+1}^{-1} + u_i u_{i+1} u_i \\
 &\in s_{i+1}^{-1}s_{i+1}^{-1}(s_i - a - bs_i^{-1})s_{i+1}^{-1} + u_i u_{i+1} u_i \\
 &\in s_{i+1}^{-1}s_{i+1}^{-1}s_i s_{i+1}^{-1} - as_{i+1}^{-1}s_{i+1}^{-1}s_{i+1}^{-1} - bs_{i+1}^{-1}(s_{i+1}^{-1}s_i^{-1}s_{i+1}^{-1}) + u_i u_{i+1} u_i \\
 &\in s_{i+1}^{-1}s_{i+1}^{-1}s_i s_{i+1}^{-1} - as_{i+1}^{-1}s_{i+1}^{-1}s_{i+1}^{-1} - bs_{i+1}^{-1}s_{i+1}^{-1}s_{i+1}^{-1} + u_i u_{i+1} u_i \\
 &\in s_{i+1}^{-1}(s_{i+1}^{-1}s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i
 \end{aligned}$$

□

Lemma 2.4. $s_{i+1}^{-1}s_i s_{i+1}^{-1} \in c^{-1}(s_{i+1}s_i^{-1}s_{i+1})s_i + u_i u_{i+1} u_i$

Proof. We have $(s_{i+1}s_i^{-1}s_{i+1})s_i = s_{i+1}(s_i^{-1}s_{i+1}s_i) = s_{i+1}s_{i+1}s_i s_{i+1}^{-1} = s_{i+1}^2 s_i s_{i+1}^{-1}$ by lemma 2.1. Since $s_{i+1}^2 = as_{i+1} + b + cs_{i+1}^{-1}$ we get $(s_{i+1}s_i^{-1}s_{i+1})s_i = (as_{i+1} + b + cs_{i+1}^{-1})s_i s_{i+1}^{-1} = as_{i+1}s_i s_{i+1}^{-1} + bs_i s_{i+1}^{-1} + cs_{i+1}^{-1}s_i s_{i+1}^{-1} \in cs_{i+1}^{-1}s_i s_{i+1}^{-1} + u_i u_{i+1} u_i$ since $s_{i+1}s_i s_{i+1}^{-1} \in u_i u_{i+1} u_i$ by lemma 2.1. □

3. THE ALGEBRA A_3

We identify A_2 with its image in A_3 under $s_i \mapsto s_i$, that is with the subalgebra of A_3 generated by s_1 . Lemma 2.1 provides the following equalities

$$\begin{aligned}
 s_2 s_1 s_2^{-1} &= s_1^{-1} s_2 s_1 \\
 s_2 s_1^{-1} s_2^{-1} &= s_1^{-1} s_2^{-1} s_1 \\
 s_2^{-1} s_1 s_2 &= s_1 s_2 s_1^{-1} \\
 s_2^{-1} s_1^{-1} s_2 &= s_1 s_2^{-1} s_1^{-1}
 \end{aligned}$$

Lemma 3.1. $s_2^{-1}s_1 s_2^{-1}A_2 \subset A_2 s_2^{-1}s_1 s_2^{-1} + u_2 u_1 u_2$ and $s_2 s_1^{-1} s_2 A_2 \subset A_2 s_2 s_1^{-1} s_2 + u_2 u_1 u_2$

Proof. Straightforward consequences of lemma 2.3 □

Theorem 3.2.

- (1) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2^{-1} s_1 s_2^{-1} A_2$
 (2) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2 s_1^{-1} s_2 A_2$
 (3) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2 s_1^{-1} s_2 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + s_2 s_1^{-1} s_2 A_2$
 (4) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2^{-1} s_1 s_2^{-1} = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + s_2^{-1} s_1 s_2^{-1} A_2$

Proof. Up to applying Φ , (2) is a consequence of (1). Then (3) and (4) are consequences of (1) and (2) by the above lemma. We now prove (1), and let U denote its RHS. It is clearly a A_2 -submodule of A_3 which contains 1, so we only need to prove $s_2 U \subset U$. Note that, clearly, $u_1 u_2 u_1 \subset U$. We first prove $u_2 u_1 u_2 \subset U$. Since we know $u_1 u_2 \subset U$, $u_2 u_1 \subset U$, this means that $w = s_2^\alpha s_1^\beta s_2^\gamma \in U$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. If α and β have opposite signs this element belongs to $u_1 u_2 u_1 \subset U$ by lemma 2.1, so we can assume $\alpha = \beta$. If $\alpha = \beta = \gamma$, then the braid relations imply $w \in u_2 u_1 u_2 \subset U$. Thus only remains $w \in \{s_2^{-1} s_1 s_2^{-1}, s_2 s_1^{-1} s_2\}$. Clearly $s_2^{-1} s_1 s_2^{-1} \in U$, and $s_2 s_1^{-1} s_2 \in c(s_2^{-1} s_1 s_2^{-1})s_1^{-1} + u_1 u_2 u_1 \subset u_1 u_2 u_1 = U$ by lemma 2.4. We thus proved $u_2 u_1 u_2 \subset U$. We now prove $s_2 U \subset U$. Clearly $s_2(A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2) \subset u_2 u_1 u_2 u_1 \subset U u_1 \subset U$, so we

need to prove $s_2 u_1 s_2^{-1} s_1 s_2^{-1} \subset U$. But $s_2 u_1 s_2^{-1} \subset u_1 u_2 u_1$ by lemma 2.1 hence $s_2 u_1 s_2^{-1} s_1 s_2^{-1} \subset u_1 u_2 u_1 u_2 \subset u_1 U \subset U$. This proves the claim. \square

Corollary 3.3. *We have $A_3 = u_1 u_2 u_1 u_2 = u_2 u_1 u_2 u_1$. Moreover,*

$$\begin{aligned} A_3 &= u_1 u_2 u_1 + u_2 u_1 u_2 + R s_1^{-1} s_2 s_1^{-1} s_2 = u_1 u_2 u_1 + u_2 u_1 u_2 + R s_2^{-1} s_1 s_2^{-1} s_1 \\ &= u_1 u_2 u_1 + u_2 u_1 u_2 + R s_1 s_2^{-1} s_1 s_2 = u_1 u_2 u_1 + u_2 u_1 u_2 + R s_2 s_1^{-1} s_2 s_1 \end{aligned}$$

Corollary 3.4. *Let $n \geq 3$. For all $1 \leq i, j \leq n-1$, we have in A_n the equality $u_i u_j u_i u_j = u_j u_i u_j u_i$.*

This theorem implies that A_3 is a free R -module of finite rank, consequently that $A_3 \subset A_3 \otimes_R K \simeq Mat_3(K) \oplus Mat_2(K)^3 \oplus K^3$ where K is a sufficiently large extension of the quotient field of R , and the isomorphism is explicitly given by the matrix models of the irreducible representations of A_3 . From this it is simply a linear algebra matter to check equalities in A_3 , or to express a given element in a given basis. We used this approach to get the following identities in A_3 .

Lemma 3.5.

$$\begin{aligned} s_2^{-1} s_1 s_2^{-1} s_1 s_2^{-1} s_1 &= \frac{-(c+ab)a}{c^2} s_1 + \frac{a}{c} s_1 s_2 + \frac{a}{c} s_1^{-1} s_2 s_1 - \frac{ab}{c} s_2^{-1} s_1 s_2^{-1} + \frac{-ab}{c} s_1^{-1} + \frac{ab}{c^2} s_2 s_1 \\ &\quad + s_1^{-1} s_2 s_1^{-1} - \frac{b}{c} s_2^{-1} s_1 s_2^{-1} s_1 - \frac{ab^2}{c^2} s_2^{-1} s_1 + \frac{b}{c} s_1^{-1} s_2 \\ &\quad - \frac{a}{c} s_1 s_2^{-1} s_1 + \frac{b}{c} s_2 s_1^{-1} - \frac{b^2}{c} s_2^{-1} s_1^{-1} - b s_1^{-1} s_2^{-1} s_1^{-1} \end{aligned}$$

Lemma 3.6.

$$\begin{aligned} s_1 s_2^{-1} s_1 s_2^{-1} &= s_2^{-1} s_1 s_2^{-1} s_1 + \frac{a}{c} s_1 s_2 - \frac{a}{c} s_2 s_1 - \frac{ab}{c} s_1 s_2^{-1} + \frac{ab}{c} s_2^{-1} s_1 + b s_2^{-1} s_1^{-1} - b s_1^{-1} s_2^{-1} \\ s_2 s_1^{-1} s_2 s_1^{-1} &= s_2^{-1} s_1 s_2^{-1} s_1 + a(s_1^{-1} s_2 s_1^{-1} - s_2^{-1} s_1 s_2^{-1}) - \frac{ab}{c} s_1 s_2^{-1} + \frac{ab}{c} s_1^{-1} s_2 + \frac{b}{c} s_1 s_2^{-1} s_1 \\ &\quad - \frac{b}{c} s_2 s_1^{-1} s_2 \\ s_1^{-1} s_2 s_1^{-1} s_2 &= s_2^{-1} s_1 s_2^{-1} s_1 + \frac{a}{c} s_1 s_2 - a s_2^{-1} s_1 s_2^{-1} - \frac{a}{c} s_2 s_1 + a s_1^{-1} s_2 s_1^{-1} \\ &\quad - \frac{ab}{c} s_1 s_2^{-1} + \frac{b}{c} s_1 s_2^{-1} s_1 + \frac{ab}{c} s_2 s_1^{-1} - \frac{b}{c} s_2 s_1^{-1} s_2 + b s_2^{-1} s_1^{-1} - b s_1^{-1} s_2^{-1} \end{aligned}$$

As a consequence, we get

Lemma 3.7.

$$\begin{aligned} s_1 s_2^{-1} s_1 s_2^{-1} - s_2^{-1} s_1 s_2^{-1} s_1 &= \frac{a}{c} s_1 s_2 - \frac{a}{c} s_2 s_1 + \frac{-ab}{c} s_1 s_2^{-1} + \frac{ab}{c} s_2^{-1} s_1 + b s_2^{-1} s_1^{-1} - b s_1^{-1} s_2^{-1} \\ s_2 s_1^{-1} s_2 s_1^{-1} - s_1^{-1} s_2 s_1^{-1} s_2 &= \frac{ab}{c} s_1^{-1} s_2 - \frac{a}{c} s_1 s_2 + \frac{a}{c} s_2 s_1 - \frac{ab}{c} s_2 s_1^{-1} - b s_2^{-1} s_1^{-1} + b s_1^{-1} s_2^{-1} \end{aligned}$$

4. THE ALGEBRA A_4 AS A A_3 (BI)MODULE

We identify A_3 with its image in A_4 , and denote $sh(A_3)$ the R -subalgebra of A_4 generated by s_2, s_3, s_4 . It is the image of A_3 under the ‘shift’ morphism $s_i \mapsto s_{i+1}$. The goal of this section is to prove the following theorem.

Theorem 4.1.

- (1) $A_4 = A_3 + A_3 s_3 A_3 + A_3 s_3^{-1} A_3 + A_3 s_3 s_2^{-1} s_3 A_3 + A_3 s_3^{-1} s_2 s_3^{-1} A_3 + A_3 s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_3$
- (2) $A_4 = A_3 + A_3 s_3 A_3 + A_3 s_3^{-1} A_3 + A_3 s_3 s_2^{-1} s_3 A_3 + A_3 s_3^{-1} s_2 s_3^{-1} + A_3 s_3 s_2^{-1} s_1 s_2^{-1} s_3$
- (3) $A_4 = A_3 + A_3 s_3 A_3 + A_3 s_3^{-1} A_3 + A_3 s_3 s_2^{-1} s_3 A_3 + s_3^{-1} s_2 s_3^{-1} s_2 s_3^{-1} A_3 + s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_3$

We denote U the right-hand side of (1). We notice that $sh(A_3) \subset U$, because of theorem 3.2 (2). Also notice that $\Psi(U) = U$ and $\Phi(U) = U$ because of lemma 2.4.

Lemma 4.2. $u_3 A_3 u_3 \subset U$.

Proof. By theorem 3.2 we have $A_3 = u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1}$ hence $u_3 A_3 u_3 \subset u_3 u_1 u_2 u_1 u_3 + u_3 u_1 s_2^{-1} s_1 s_2^{-1} u_3$. But $u_3 u_1 u_2 u_1 u_3 = u_1 u_3 u_2 u_3 u_1 \subset u_1 sh(A_3) u_1 \subset u_1 U u_1 \subset U$, and $u_3 u_1 s_2^{-1} s_1 s_2^{-1} u_3 = u_1 u_3 s_2^{-1} s_1 s_2^{-1} u_3$ so we need to prove $s_3^\alpha s_2^{-1} s_1 s_2^{-1} s_3^\beta \in U$ for $\alpha, \beta \in \{-1, 1\}$. The case $(\alpha, \beta) = (1, 1)$ is clear by definition of U . When $(\alpha, \beta) = (-1, -1)$, we have $s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} \in c^{-1} s_3^{-1} s_2 s_1^{-1} s_2 s_1 s_3^{-1} + s_3^{-1} u_1 u_2 u_1 s_3^{-1}$ that is $s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} \in s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} u_1 + u_1 s_3^{-1} u_2 s_3^{-1} u_1 \subset U + u_1 sh(A_3) u_1 \subset U$. When $(\alpha, \beta) = (1, -1)$ we get $s_3 s_2^{-1} s_1 s_2^{-1} s_3^{-1} = s_3 s_2^{-1} s_1 (s_2^{-1} s_3^{-1} s_2^{-1}) s_2 = s_3 s_2^{-1} s_1 s_3^{-1} s_2^{-1} s_3^{-1} s_2 = (s_3 s_2^{-1} s_3^{-1}) s_1 s_2^{-1} s_3^{-1} s_2 = s_2^{-1} s_3^{-1} (s_2 s_1 s_2^{-1}) s_3^{-1} s_2 \in s_2^{-1} s_3^{-1} u_1 u_2 u_1 s_3^{-1} s_2 \subset s_2^{-1} u_1 s_3^{-1} u_2 s_3^{-1} u_1 s_2 \subset A_3 sh(A_3) A_3 \subset U$. The case $(-1, 1)$ is similar. \square

Lemma 4.3. $u_3 A_3 u_3 A_3 u_3 \subset A_3 u_3 A_3 u_3 A_3$

Proof. For $x \in A_3$, we say that x has at most p factors if it belongs to $u_{\sigma(1)} \dots u_{\sigma(p)}$ for some $\sigma : [1, p] \rightarrow \{1, 2\}$. By theorem 3.2 the minimal number of factors for such an x is at most 4. We let $x, y \in A_3$, with minimal number of factors p and q , and prove that $u_3 x u_3 y u_3 \subset A_3 u_3 A_3 u_3 A_3$ by induction on (p, q) in lexicographic order. Note that, since $\Psi(U) = U$, we may assume $p \geq q$. Moreover, since $A_3 = u_1 u_2 u_1 u_2$ and $u_3(u_1 u_2 u_1 u_2) u_3 y u_3 = u_1 u_3 u_2 u_1 u_2 u_3 y u_3$, we can assume $p \leq 3$ (hence $q \leq 3$).

The case $q = 0$ is trivial. If $x \in u_1 u_{\sigma(2)} \dots u_{\sigma(p)}$, we have $u_3 x u_3 y u_3 \in u_3 u_1 u_{\sigma(2)} \dots u_{\sigma(p)} u_3 y u_3 = u_1 u_3 u_{\sigma(2)} \dots u_{\sigma(p)} u_3 y u_3$ and we are reduced to the case $(p-1, q)$. Similarly, if $y \in u_{\tau(1)} \dots u_{\tau(q-1)} u_1$, we are reduced to $(p, q-1)$. As a consequence, the only non-trivial case for $p \leq 1$ is $u_3 u_2 u_3 u_2 u_3 \subset sh(A_3) \subset A_3 u_3 A_3 u_3 A_3$ because of theorem 3.2.

We consider the case $(p, q) = (2, 1)$. The only nontrivial case is $u_3 u_2 u_1 u_3 u_2 u_3$. We need to prove $s_3^\alpha u_2 u_1 s_3^\beta u_2 s_3^\gamma \subset A_3 u_3 A_3 u_3 A_3$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. Because $s_3^\alpha u_2 u_1 (s_3^\beta u_2 s_3^\gamma) = (s_3^\alpha u_2 s_3^\beta) u_1 u_2 s_3^\gamma$ this is clear by lemma 2.1 unless α, β and γ are all the same. We thus need to prove $s_3^\alpha s_2^\beta u_1 s_3^\alpha s_2^\gamma s_3^\alpha \subset A_3 u_3 A_3 u_3 A_3$ for all $\beta, \gamma \in \{-1, 1\}$. Since $s_3^\alpha s_2^\beta s_3^\alpha = s_2^\beta s_3^\alpha s_2^\beta$ we can assume $\beta = -\alpha$ and $\gamma = -\alpha$ and consider $s_3^\alpha s_2^{-\alpha} u_1 s_3^\alpha s_2^{-\alpha} s_3^\alpha$. By lemma 2.4 we have $s_3^\alpha s_2^{-\alpha} s_3^\alpha \in s_3^{-\alpha} s_2^\alpha s_3^{-\alpha} u_2 + u_2 u_3 u_2$ hence $s_3^\alpha s_2^{-\alpha} u_1 s_3^\alpha s_2^{-\alpha} s_3^\alpha \subset s_3^\alpha s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^\alpha s_3^{-\alpha} u_2 + s_3^\alpha s_2^{-\alpha} u_1 u_2 u_3 u_2 \subset s_3^\alpha s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^\alpha s_3^{-\alpha} u_2 + u_3 A_3 u_3 A_3$ and we already noticed

$$s_3^\alpha s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^\alpha s_3^{-\alpha} u_2 = (s_3^\alpha s_2^{-\alpha} s_3^{-\alpha}) u_1 s_2^\alpha s_3^{-\alpha} u_2 \subset u_2 u_3 u_2 u_1 u_2 u_3 u_2 \subset A_3 u_3 A_3 u_3 A_3.$$

All cases for $(p, q) = (2, 2)$ can be easily reduced to smaller values by commutation relations. The only a priori irreducible case for $(p, q) = (3, 1)$ is $u_3 u_2 u_1 u_2 u_3 u_2 u_3$. Since $u_2 u_3 u_2 u_3 \subset u_2 u_3 u_2 + u_3 u_2 u_3$ by theorem 3.2, we are reduced to case $(2, 1)$.

For the case $(p, q) = (3, 2)$, we can use a similar argument : the only nontrivial case is $u_3 u_2 u_1 u_2 u_3 u_1 u_2 u_3 = u_3 u_2 u_1 u_2 u_1 u_3 u_2 u_3$ and $u_2 u_1 u_2 u_1 \subset u_2 u_1 u_2 + u_1 u_2 u_1$, and we are reduced to smaller cases.

The only remaining case is thus $(p, q) = (3, 3)$. Since $x \in A_3 = u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1}$ and $y \in A_3 = u_1 u_2 u_1 + s_2^{-1} s_1 s_2^{-1} u_1$ we are reduced to considering $s_3^\alpha s_2^{-1} s_1 s_2^{-1} s_3^\beta s_2^{-1} s_1 s_2^{-1} s_3^\gamma$ for $\alpha, \beta, \gamma \in \{-1, 1\}$. Up to applying Φ if necessary, we can assume $\beta = -1$. Then $s_3^\alpha s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3^\gamma = s_3^\alpha s_2^{-1} s_1 s_3^{-1} s_2^{-1} s_3^{-1} s_1 s_2^{-1} s_3^\gamma = (s_3^\alpha s_2^{-1} s_3^{-1}) s_1 s_2^{-1} s_1 (s_3^{-1} s_2^{-1} s_3^\gamma) \subset u_2 u_3 u_2 A_3 u_2 u_3 u_2 \subset A_3 u_3 A_3 u_3 A_3$ by lemma 2.2, and this concludes the proof. \square

We let $U_0 = A_3 u_3 A_3 + A_3 s_3 s_2^{-1} s_3 A_3 = A_3 u_3 A_3 + A_3 s_3^{-1} s_2 s_3^{-1} A_3 = A_3 sh(A_3) A_3 \subset U$.

Lemma 4.4.

- (1) $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} A_3 \subset A_3 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$
- (2) $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} A_2 \subset A_2 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$
- (3) $s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_3 \subset A_3 s_3 s_2^{-1} s_1 s_2^{-1} s_3 + U_0$
- (4) $s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_2 \subset A_2 s_3 s_2^{-1} s_1 s_2^{-1} s_3 + U_0$

Statements (3) and (4) are consequences of (1) and (2) by application of Φ , and (1) and (2) are immediate consequences of the more detailed lemma below.

Lemma 4.5.

- (1) For all $x \in A_2$, $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} x \in x s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$.
- (2) $(s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) s_2^{-1} \in s_1^{-1} s_2^{-1} s_1 (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0$
- (3) For all $x \in A_3$, $(s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) x \in x^{s_1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0$ (where $x^{s_1} = s_1^{-1} x s_1$).

Proof. We first prove (1). We have

$$\begin{aligned} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) s_1^{-1} &= s_3^{-1} (s_2 s_1^{-1} s_2) s_1^{-1} s_3^{-1} \\ &\in s_3^{-1} s_1^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} + s_3^{-1} u_1 u_2 u_1 s_3^{-1} \\ &\subset s_3^{-1} s_1^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} + u_1 s_3^{-1} u_2 s_3^{-1} u_1 \\ &\subset s_1^{-1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + A_3 sh(A_3) A_3 \\ &\subset s_1^{-1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0 \end{aligned}$$

by lemma 2.3. Since s_1^{-1} generates A_2 this proves (1).

We now prove (2). We have

$$\begin{aligned} (s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} &= s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1}s_2^{-1} \\ &\in cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1}s_2^{-1} \\ &\subset cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + U_0 \end{aligned}$$

by lemma 2.4. By lemma 2.3 it follows that

$$\begin{aligned} &(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} && \in cs_3^{-1}s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_2^{-1} + U_0 \\ = &cs_1^{-1}s_3^{-1}s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2^{-1}) + U_0 && = cs_1^{-1}s_3^{-1}s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1} + U_0 \\ = &cs_1^{-1}(s_3^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_3^{-1}) + U_0 && = cs_1^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} + U_0 \\ \subset &cs_1^{-1}s_2^{-1}s_3^{-1}c^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0 && = s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0 \end{aligned}$$

again by lemma 2.4. Since

$$s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} \in s_1^{-1}s_2^{-1}s_3^{-1}s_1(s_2s_1^{-1}s_2)s_3^{-1} + U_0 \subset s_1^{-1}s_2^{-1}s_1s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1} + U_0$$

by lemma 2.3, this proves (2)

Since A_3 is generated by s_1^{-1} and s_2^{-1} and U_0 is a $A_3 - A_3$ submodule of A_4 , we need to check (3) only for $x = s_1^{-1}$ and $x = s_2^{-1}$, and we just did. \square

Proof of theorem 4.1.

Since $1 \in U$ and U is a A_3 -submodule of A_4 , in order to prove (1) one need to prove $s_3U \subset U$. Clearly $U \subset A_3u_3A_3u_3A_3$ hence $s_3U \subset u_3A_3u_3A_3u_3A_3 \subset A_3u_3A_3u_3A_3$ by lemma 4.3, and $A_3(u_3A_3u_3)A_3 \subset A_3UA_3 = U$ by lemma 4.2 which proves the claim. Then (2) and (3) are consequences of (1) by lemma 4.4. This concludes the proof of the theorem.

We now let $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, and $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} \in A_4$. We recall that $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3 \subset A_4$ is a sub-bimodule, and let $U^+ = A_3w^+ + U_0$.

Let $w_0 = s_3s_2s_1^2s_2s_3$. It is classical that, already in the braid group B_4 , w_0 commutes with s_1 and s_2 . Thus clearly $A_3w_0A_3 = A_3w_0$ and $A_3w_0^{-1}A_3 = A_3w_0^{-1}$. The lemma below thus provides another explanation of lemma 4.4 above.

Lemma 4.6.

- (1) $w_0 \in A_3^\times w^+ + U_0$, $w_0^{-1} \in A_3^\times w^- + U_0$
- (2) $U^+ = A_3w_0 + U_0$
- (3) $s_3A_3s_3^{-1} \subset U_0$, $s_3^{-1}A_3s_3 \subset U_0$
- (4) $s_3A_3s_3 \subset U^+$

Proof. We have $w_0 = s_3(s_2s_1^2s_2)s_3 \in cs_3s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1s_2s_3 + Rs_3s_2^2s_3$. Clearly $s_3s_2^2s_3 \in U_0$ and $s_3(s_2s_1s_2)s_3 = s_3(s_1s_2s_1)s_3 = s_1s_3s_2s_3s_1 \in U_0$. Moreover, by lemmas 2.4 and 2.3, $s_3(s_2s_1^{-1}s_2)s_3 \in cs_3s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3 + s_3u_1u_2u_1s_3 \subset cs_1^{-1}w^+ + U_0$ and thus $w_0 \in A_3^\times w^+ + U_0$. As a consequence, $w_0^{-1} = s_3^{-1}s_2^{-1}s_1^{-2}s_2^{-1}s_3^{-1} = \Phi(w_0) \in \Phi(A_3^\times)\Phi(w^+) + \Phi(U_0) = A_3^\times w^- + U_0$, and this proves (1). By definition we have $U^+ = A_3w^+ + U_0 \subset A_3(A_3^\times w_0 + U_0) + U_0 \subset A_3w_0 + U_0$, and conversely $A_3w_0 + U_0 \subset A_3(A_3^\times w^+ + U_0) + U_0 \subset U^+$; this proves (2). (3) and (4) are given by the proof of lemma 4.2. \square

An immediate consequence is the following variation on theorem 4.1.

Theorem 4.7.

- (1) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0A_3 + A_3w_0^{-1}A_3$
- (2) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0 + A_3w_0^{-1}$
- (3) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + w_0A_3 + w_0^{-1}A_3$

From this one easily gets the following generating set of A_4 as A_3 -module. Another generating set can be found in [3] §4B.

Proposition 4.8. *As a left A_3 -module, A_4 is generated by the 27 elements*

$$\{1, s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}, s_3s_2^{-1}s_1s_2^{-1}s_3, s_3, s_3^{-1}, s_3^{\pm}s_2^{\pm}, s_3^{\pm}s_2^{\pm}s_1^{\pm},$$

$$s_3^{\pm}s_2^{-1}s_1s_2^{-1}, s_3s_2^{-1}s_3, s_3s_2^{-1}s_3s_1^{\pm}, s_3s_2^{-1}s_3s_1s_2^{-1}s_1, s_3s_2^{-1}s_3s_1^{\pm}s_2^{\pm}\}.$$

Proof. We denote S the set of 27 elements of the statement and L its span as a A_3 -module. We have $A_4 = A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3$ by theorem 4.1, and clearly $A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 \subset L$. Moreover, since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2^{-1}s_1s_2^{-1}$ we have

$$A_3s_3^{\alpha}A_3 = A_3s_3^{\alpha} + \sum_{\substack{\varepsilon \in \{-1, 0, 1\} \\ \beta \in \{-1, 1\}}} A_3s_3^{\alpha}s_2^{\beta}s_1^{\varepsilon} + A_3s_3^{\alpha}s_2^{-1}s_1s_2^{-1} \subset L$$

for any $\alpha \in \{-1, 1\}$. It remains to prove $A_3s_3s_2^{-1}s_3A_3 \subset L$. Since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + s_2^{-1}s_1s_2^{-1}A_2$, we have $A_3s_3s_2^{-1}s_3A_3 = A_3s_3s_2^{-1}s_3A_2 + A_3s_3s_2^{-1}s_3A_2s_2^{-1}A_2 + A_3s_3s_2^{-1}s_3A_2s_2A_2 + A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2$. Clearly $A_3s_3s_2^{-1}s_3A_2$ is A_3 -spanned by the $s_3s_2^{-1}s_3s_1^{\varepsilon}$ for $\varepsilon \in \{0, 1, -1\}$ hence $A_3s_3s_2^{-1}s_3A_2 \subset L$. Now $s_3s_2^{-1}s_3s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3 + u_2u_3 + u_3u_2$ by lemma 3.6, hence $A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3u_2s_1s_2^{-1}A_2 + A_3u_3s_1s_2^{-1}A_2$, that is

$$A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3A_3.$$

We thus only need to show $A_3s_3s_2^{-1}s_3A_2s_2^{\beta}A_2 \subset L$ for $\beta \in \{-1, 1\}$. This module is A_3 -spanned by the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{\gamma}$ for $\alpha, \gamma \in \{0, 1, -1\}$. The elements belong to S when $\gamma = 0$, so we can assume $\gamma \in \{-1, 1\}$. When $\alpha = 0$, in case $\beta = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1^{\gamma} = s_3s_3s_2s_3^{-1}s_1^{\gamma} \in u_3s_2s_3^{-1}s_1^{\gamma}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\gamma}$ for $\varepsilon \in \{-1, 0, 1\}$. In case $\varepsilon = 0$ such an element belongs to $A_3u_3A_3 \subset L$; when $\varepsilon = 1$ we can use $(s_3s_2s_3^{-1})s_1^{\gamma} = s_2^{-1}s_3s_2s_1^{\gamma} \in A_3s_3s_2s_1^{\gamma} \in L$; when $\varepsilon = -1$ we have $s_3^{-1}s_2s_3^{-1}s_1^{\gamma} \in A_3s_3s_2^{-1}s_3s_1^{\gamma}$ by lemmas 2.4 and 2.3, and $s_3s_2^{-1}s_3s_1^{\gamma} \in S$. We can thus assume $\alpha \neq 0$.

We consider first the case $\gamma = -\alpha$. We have $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{-\alpha} = s_3s_2^{-1}s_3s_2^{-\alpha}s_1^{\beta}s_2^{\alpha}$. Then, either $\alpha = 1$ and, by lemma 3.6,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{\beta}s_2 \in s_2^{-1}s_3s_2^{-1}s_3s_1^{\beta}s_2 + u_2u_3s_1^{\beta}s_2 + u_3u_2s_1^{\beta}s_2 \subset L,$$

or $\alpha = -1$ and $s_3(s_2^{-1}s_3s_2)s_1^{\beta}s_2^{-1} = s_3s_3s_2s_3^{-1}s_1^{\beta}s_2^{-1} \in u_3s_2s_3^{-1}s_1^{\beta}s_2^{-1}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\beta}s_2^{-1}$ which clearly belong to L for $\varepsilon = 0$ and, because of $s_3s_2s_3^{-1} = s_2^{-1}s_3s_2$, for $\varepsilon = 1$; in case $\varepsilon = -1$ it is readily shown to belong to L by lemmas 2.4 and 2.3 applied to $s_3^{-1}s_2s_3^{-1}$.

We can now assume $\gamma = \alpha$. In case $\beta = \alpha = \gamma$, we have $s_3s_2^{-1}s_3(s_1^{\alpha}s_2^{\alpha}s_1^{\alpha}) = s_3s_2^{-1}s_3s_2^{\alpha}s_1^{\alpha}s_2^{\alpha}$ and, when $\alpha = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1s_2 = s_3s_3s_2s_3^{-1}s_1s_2 \in u_3s_2s_3^{-1}s_1s_2 \subset L$ by similar arguments as for $u_3s_2s_3^{-1}s_1s_2^{-1}$; when $\alpha = -1$, we have, by lemma 3.6,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2^{-1} + u_3u_2s_1^{-1}s_2^{-1} + u_2u_3s_1^{-1}s_2^{-1} \subset L.$$

We thus only need to consider the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}$. By lemmas 2.4 and 2.3, we have $s_1^{-1}s_2s_1^{-1} \in s_2(s_1s_2^{-1}s_1) + u_2u_1u_2$, and $s_3s_2^{-1}s_3u_2u_1u_2$ belongs to the A_3 -span of our list by our previous arguments. It follows that it only remains to consider $s_3s_2^{-1}s_3s_1s_2^{-1}s_1$, which belongs to our list, and $s_3(s_2^{-1}s_3s_2)s_1s_2^{-1}s_1 = s_3^2s_2s_3^{-1}s_1s_2^{-1}s_1$, which lies in the linear span of the $s_3^{\varepsilon}s_2s_3^{-1}s_1s_2^{-1}s_1$ for $\varepsilon \in \{-1, 0, 1\}$. Clearly this element belongs to L in case $\varepsilon = 0$, when $\varepsilon = 1$ it also belongs to L because of $(s_3s_2s_3^{-1})s_1s_2^{-1}s_1 = s_2^{-1}s_3s_2s_1s_2^{-1}s_1 \in A_3u_3A_3 \subset L$, and when $\varepsilon = -1$ lemmas 2.4 and 2.3 applied to $s_3^{-1}s_2s_3^{-1}$ show that

$$s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_1 \in A_3s_3s_2^{-1}s_3s_1s_2^{-1}s_1 + A_3u_3A_3 \subset L,$$

and this concludes the proof. \square

For subsequent use we prove here the following lemma.

Lemma 4.9. $w_0^2 \in A_3^{\times}w_0^{-1} + U^+$.

Proof. We have $w_0^2 = s_3 s_2 (s_1^2) s_2 s_3^2 s_2 s_1^2 s_2 s_3 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_1^2 s_2 s_3 + R s_3 s_2 s_1 s_2 s_3^2 s_2 s_1^2 s_2 s_3 + R s_3 s_2^2 s_3^2 s_2 s_1^2 s_2 s_3$, and $s_3 (s_2 s_1 s_2) s_3^2 s_2 s_1^2 s_2 s_3 = s_3 s_1 s_2 s_1 s_2^2 s_2 s_1^2 s_2 s_3 = s_1 (s_3 s_2 s_3) s_3 s_1 s_2 s_1^2 s_2 s_3 = s_1 s_2 (s_3 s_2 s_3) s_1 s_2 s_1^2 s_2 s_3 = s_1 s_2 s_2 s_3 s_2 s_1 s_2 s_1^2 s_2 s_3 \in U_0^+$ by lemma 4.6, while $s_3 s_2^2 (s_3^2) s_2 s_1^2 s_2 s_3 \in R s_3 s_2^2 s_3^{-1} s_2 s_1^2 s_2 s_3 + R s_3 s_2^2 s_3 s_2 s_1^2 s_2 s_3 + R s_3 s_3^2 s_2 s_1^2 s_2 s_3$, clearly $s_3 s_2^3 s_2 s_1^2 s_2 s_3 \in U^+$ by lemma 4.6,

$$\begin{aligned} s_3 s_2^2 s_3 s_2 s_1^2 s_2 s_3 &= s_3 s_2 (s_2 s_3 s_2) s_1^2 s_2 s_3 \\ &= s_3 s_2 s_3 s_2 s_3 s_1^2 s_2 s_3 = (s_3 s_2 s_3) s_2 s_1^2 (s_3 s_2 s_3) \\ &= s_2 s_3 s_2 s_2 s_1^2 s_2 s_3 s_2 \in U^+ \end{aligned}$$

by lemma 4.6, and finally

$$(s_3 s_2^2 s_3^{-1}) s_2 s_1^2 s_2 s_3 = s_2^{-1} (s_3^2) s_2^2 s_1^2 s_2 s_3 \in R s_2^{-1} s_3^{-1} s_2^2 s_1^2 s_2 s_3 + R s_2^{-1} s_3 s_2^2 s_1^2 s_2 s_3 + R s_2 s_1^2 s_2 s_3 \subset U^+$$

by lemma 4.6.

Thus, $w_0^2 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_1^2 s_2 s_3 + U^+$. Now, $s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 (s_1^2) s_2 s_3 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_1^{-1} s_2 s_3 + R s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_1 s_2 s_3 + R s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_3$. We have $s_3 s_2 s_1^{-1} s_2 s_3^2 (s_2 s_1 s_2) s_3 = s_3 s_2 s_1^{-1} s_2 s_3^2 s_1 s_2 s_1 s_3 = s_3 s_2 (s_1^{-1} s_2 s_1) s_3^2 s_2 s_3 s_1 = s_3 s_2^2 s_1 (s_2^{-1} s_3^2 s_2) s_3 s_1 = s_3 s_2^2 s_1 s_3 s_2^2 s_3^{-1} s_3 s_1 = s_3 s_2^2 s_1 s_3 s_2^2 s_1 \in U^+$ by lemma 4.6. Now $s_3 s_2 s_1^{-1} s_2 (s_3^2) s_2^2 s_3 \in R s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_2^2 s_3 + R s_3 s_2 s_1^{-1} s_2 s_3 s_2^2 s_3 + R s_3 s_2 s_1^{-1} s_3^2 s_3$; we have $s_3 s_2 s_1^{-1} s_3^2 s_3 \in U^+$ by lemma 4.6, $s_3 s_2 s_1^{-1} s_2 s_3 s_2^2 s_3 = s_3 s_2 s_1^{-1} (s_2 s_3 s_2) s_2 s_3 = s_3 s_2 s_1^{-1} s_3 s_2 s_3 s_2 s_3 = (s_3 s_2 s_3) s_1^{-1} s_2 (s_3 s_2 s_3) = s_2 s_3 s_2 s_1^{-1} s_2 s_2 s_3 s_2 \in U^+$ by lemma 4.6, and $s_3 s_2 s_1^{-1} s_2 (s_3^{-1} s_2^2 s_3) = s_3 s_2 s_1^{-1} s_2 s_2 s_3^2 s_2^{-1} \in s_3 s_2 s_1^{-1} s_2^2 s_2 (R + R s_3 + R s_3^{-1}) s_2^{-1} \subset U^+$ by lemma 4.6.

Thus $w_0^2 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^2 s_2 s_1^{-1} s_2 s_3 + U^+$. Now,

$$s_3 s_2 s_1^{-1} s_2 (s_3^2) s_2 s_1^{-1} s_2 s_3 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_2 s_1^{-1} s_2 s_3 + R s_3 s_2 s_1^{-1} s_2 s_3 s_2 s_1^{-1} s_2 s_3 + R s_3 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2 s_3.$$

We have

$$\begin{aligned} s_3 s_2 s_1^{-1} (s_2 s_3 s_2) s_1^{-1} s_2 s_3 &= s_3 s_2 s_1^{-1} s_3 s_2 s_3 s_1^{-1} s_2 s_3 = (s_3 s_2 s_3) s_1^{-1} s_2 s_1^{-1} (s_3 s_2 s_3) \\ &= s_2 s_3 s_2 s_1^{-1} s_2 s_1^{-1} s_2 s_3 s_2 \in U^+ \end{aligned}$$

by lemma 4.6, $s_3 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2 s_3 \in U^+$ by lemma 4.6, hence $w_0^2 \in R^\times s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_2 s_1^{-1} s_2 s_3 + U^+$. Using $s_2 s_1^{-1} s_2 \in A_2^\times s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ (see lemmas 2.4 and 2.3), we get $s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_2 s_1^{-1} s_2 s_3 \in A_2^\times s_3 s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_2 s_1^{-1} s_2 s_3 + s_3 u_1 u_2 u_1 s_3^{-1} s_2 s_1^{-1} s_2 s_3$. Since

$$s_3 u_1 u_2 u_1 s_3^{-1} s_2 s_1^{-1} s_2 s_3 = u_1 (s_3 u_2 s_3^{-1}) u_1 s_2 s_1^{-1} s_2 s_3 \subset u_1 s_2^{-1} u_3 s_2 u_1 s_2 s_1^{-1} s_2 s_3 \subset U^+$$

by lemma 4.6, we have $w_0^2 \in A_2^\times s_3 s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_2 s_1^{-1} s_2 s_3 + U^+$. Now

$$\begin{aligned} s_3 s_2^{-1} s_1 (s_2^{-1} s_3^{-1} s_2) s_1^{-1} s_2 s_3 &= s_3 s_2^{-1} s_1 s_3 s_2^{-1} s_3^{-1} s_1^{-1} s_2 s_3 = s_3 s_2^{-1} s_1 s_3 s_2^{-1} s_1^{-1} (s_3^{-1} s_2 s_3) \\ &= s_3 s_2^{-1} s_1 s_3 (s_2^{-1} s_1^{-1} s_2) s_3 s_2^{-1} = s_3 s_2^{-1} s_1 s_3 s_1 s_2^{-1} s_1^{-1} s_3 s_2^{-1} = s_3 s_2^{-1} s_3 s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1} \end{aligned}$$

and, using $s_3 s_2^{-1} s_3 \in u_2^\times s_3^{-1} s_2 s_3^{-1} + u_2 u_3 u_2$, we get

$$s_3 s_2^{-1} s_3 s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1} \in u_2^\times s_3^{-1} s_2 s_3^{-1} s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1} + u_2 u_3 u_2 s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1}.$$

We have $u_2 u_3 u_2 s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1} \in U^+$ by lemma 4.6, and

$$s_3^{-1} s_2 s_3^{-1} s_1^2 s_2^{-1} s_1^{-1} s_3 s_2^{-1} = s_3^{-1} s_2 s_1^2 (s_3^{-1} s_2^{-1} s_3) s_1^{-1} s_2^{-1} = s_3^{-1} s_2 s_1^2 s_2 s_3^{-1} s_2^{-1} s_1^{-1} s_2^{-1}.$$

Thus $w_0^2 \in A_3^\times s_3^{-1} s_2 s_1^2 s_2 s_3^{-1} (s_2^{-1} s_1^{-1} s_2^{-1}) + U^+$. Since $s_3^{-1} s_2 (s_1^2) s_2 s_3^{-1} \in R^\times s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + R s_3^{-1} s_2 s_1 s_2 s_3^{-1} + R s_3^{-1} s_2^2 s_3^{-1}$ and clearly $s_3^{-1} s_2^2 s_3^{-1} \in U_0$, $s_3^{-1} (s_2 s_1 s_2) s_3^{-1} = s_3^{-1} s_1 s_2 s_1 s_3^{-1} = s_1 s_3^{-1} s_2 s_3^{-1} s_1 \in U_0$, we have $w_0^2 \in A_3^\times w_0^{-1} (s_2^{-1} s_1^{-1} s_2^{-1}) + U^+$, hence $w_0^2 \in A_3^\times w_0^{-1} (s_2^{-1} s_1^{-1} s_2^{-1}) + U^+$ by lemma 4.6 (1). Since w_0 commutes with s_1 and s_2 this yields $w_0^2 \in A_3^\times w_0^{-1} + U^+$. \square

5. THE ALGEBRA A_4 AS A $\langle s_1, s_3 \rangle$ (BI)MODULE

Let $B = \langle s_1, s_3 \rangle$ denote the subalgebra (with 1) of A_4 generated by s_1 and s_3 . In order to describe A_5 as a A_4 -module we will need the description of A_4 as a B -module, that we do in this section. Note that this will provide another proof of the conjecture of [4] for A_4 .

First note that there are *three* automorphisms or skew-automorphisms of the pair (A_4, B) : in addition to the automorphism Φ and the skew-automorphism Ψ , there is the automorphism $\text{Ad } \Delta : x \mapsto \Delta x \Delta^{-1}$, where Δ is the image in A_4 of Garside's Δ in the braid group on 4 strands, that is $\Delta = s_1 s_2 s_3 s_1 s_2 s_1 = s_1 (s_2 s_3 s_1 s_2) s_1$; this automorphism exchanges s_1 and s_3 and fixes s_2 .

We denote $A_4^{[0]} = B$, $A_4^{[n+1]} = A_4^{[n]}u_2B = A_4^{[n]} + A_4^{[n]}s_2B + A_4^{[n]}s_2^{-1}B$, and in particular $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$.

We first prove several lemmas.

Lemma 5.1.

- (1) For $i, j \in \{1, 3\}$ we have $u_2u_iu_2u_ju_2 \subset A_4^{[2]}$.
- (2) For $i, j, k \in \{1, 3\}$ we have $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$ and $u_2u_iu_ju_2u_ku_2 \subset A_4^{[2]}$

Proof. We prove (1). If $i = j$, up to applying $\text{Ad } \Delta$ we can assume $i = j = 1$ and the statement is a consequence of the study of A_3 , as $u_2u_1u_2u_1u_2 \subset A_3 \subset u_1u_2u_1u_2 + u_1u_2u_1$. Thus we can assume $i \neq j$, and by using $\text{Ad } \Delta$ and Ψ we only need to consider $X = s_2^\alpha s_1^\beta s_2^\gamma s_3^\delta s_2^\varepsilon$ with $\alpha, \dots, \varepsilon \in \{-1, 1\}$. If $\alpha = -\gamma$ or $\gamma = -\varepsilon$, then we get $X \in A_4^{[2]}$ by using $s_2^\alpha s_1^\beta s_2^{-\alpha} = s_1^{-\alpha} s_2^\beta s_1^\alpha$ and $s_2^\gamma s_3^\delta s_2^{-\gamma} = s_3^{-\gamma} s_2^\delta s_3^\gamma$. Up to applying Φ we can thus assume $\alpha = \gamma = \varepsilon = 1$, that is $X = s_2 s_1^\beta s_2 s_3^\delta s_2$. If $\beta = 1$ or $\delta = 1$ we get $X \in A_4^{[2]}$ by $s_2 s_1 s_2 = s_1 s_2 s_1$ and $s_2 s_3 s_2 = s_3 s_2 s_3$. One can thus assume $X = s_2 s_1^{-1} s_2 s_3^{-1} s_2$. By lemmas 2.4 and 2.3 we have $s_2 s_1^{-1} s_2 \in u_1^\times s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ hence $X \in u_1^\times s_2^{-1} s_1 s_2^{-1} s_3 s_2 + A_4^{[2]}$ and $s_2^{-1} s_1 (s_2^{-1} s_3 s_2) = s_2^{-1} s_1 s_3 s_2 s_3^{-1} \in A_4^{[2]}$, and this concludes the proof of (1).

We prove (2). Up to applying Ψ we can confine ourselves to prove $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$. By (1) and $u_j^2 = u_j$, $u_k^2 = u_k$ we can assume $j \neq k$, that is $\{j, k\} = \{1, 3\}$. Up to applying $\text{Ad } \Delta$ we can assume $i = 1$, hence we want to prove $u_2u_1u_2u_1u_3u_2 \subset A_4^{[2]}$. We have $u_2u_1u_2u_1 \subset A_3 = u_1u_2u_1u_2 + u_1u_2u_1$ hence $u_2u_1u_2u_1u_3u_2 \subset u_1u_2u_1u_2u_3u_2 + u_1u_2u_1u_3u_2 \subset A_4^{[2]}$ by (1). \square

Lemma 5.2.

$$A_4^{[3]} \subset A_4^{[2]} + \sum_{\alpha, \beta \in \{-1, 1\}} Bs_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha B + \sum_{\alpha, \beta \in \{-1, 1\}} Bs_2^\alpha (s_1 s_3)^\beta s_2^{-\beta} (s_1 s_3)^\beta s_2^\varepsilon B$$

Proof. We only need to prove that all the terms of the form $s_2^\alpha s_1^{\beta_1} s_3^{\beta_3} s_2^\gamma s_1^{\delta_1} s_3^{\delta_3} s_2^\varepsilon$ belong to the RHS, as all the over natural linear generators of $A_4^{[3]}$ belong to $A_4^{[2]}$ by lemma 5.1. We remark that the RHS is stable under Φ , Ψ and $\text{Ad } \Delta$.

We first assume $\beta_1 = -\delta_1$. Then

$$s_2^\alpha s_1^{\beta_1} s_3^{\beta_3} s_2^\gamma s_1^{\delta_1} s_3^{\delta_3} s_2^\varepsilon = s_2^\alpha s_3^{\beta_3} (s_1^{\beta_1} s_2^\gamma s_1^{-\beta_1}) s_3^{\delta_3} s_2^\varepsilon = s_2^\alpha s_3^{\beta_3} s_2^{-\beta_1} s_1^\gamma s_2^{\beta_1} s_3^{\delta_3} s_2^\varepsilon.$$

If $\alpha = \beta_1$ or $\varepsilon = -\beta_1$, such a term belongs to $A_4^{[2]}$ by $s_2^\alpha s_3^{\beta_3} s_2^{-\alpha} = s_3^{-\alpha} s_2^{\beta_3} s_3^\alpha$ or $s_2^{-\varepsilon} s_3^{\delta_3} s_2^\varepsilon = s_3^\varepsilon s_2^{\delta_3} s_3^{-\varepsilon}$ and lemma 5.1. We thus only need to consider $X = s_2^{-\beta_1} s_3^{\beta_3} s_2^{-\beta_1} s_1^\gamma s_2^{\beta_1} s_3^{\delta_3} s_2^{\beta_1}$. Since $s_2^{-\beta_1} s_3^{-\beta_1} s_2^{-\beta_1} = s_3^{-\beta_1} s_2^{-\beta_1} s_3^{-\beta_1}$ and $s_2^{\beta_1} s_3^{\beta_1} s_2^{\beta_1} = s_3^{\beta_1} s_2^{\beta_1} s_3^{\beta_1}$, by lemma 5.1 we can assume $\beta_3 = \beta_1$ and $\delta = -\beta_1$, that is $X = s_2^{-\beta_1} s_3^{\beta_1} s_2^{-\beta_1} s_1^\gamma s_2^{\beta_1} s_3^{-\beta_1} s_2^{\beta_1}$. By applying Φ and Ψ we can assume $X = s_2 s_3^{-1} s_2 s_1 s_2^{-1} s_3 s_2^{-1}$. By lemma 2.4 we have $s_2^{-1} s_3 s_2^{-1} \in s_2 s_3^{-1} s_2 u_3^\times + u_3 u_2 u_3$ hence $s_2 s_3^{-1} s_2 s_1 (s_2^{-1} s_3 s_2^{-1}) \in s_2 s_3^{-1} s_2 s_1 s_2 s_3^{-1} s_2 u_3^\times + s_2 s_3^{-1} s_2 s_1 u_3 u_2 u_3$. We have $s_2 s_3^{-1} s_2 s_1 u_3 u_2 u_3 \subset A_4^{[2]}$ by lemma 5.1 and $s_2 s_3^{-1} s_2 s_1 s_2 s_3^{-1} s_2$ belongs to the RHS, which concludes this case.

The case $\beta_3 = -\delta_3$ is a consequence of the previous case by applying $\text{Ad } \Delta$. We thus only need to consider $X = s_2^\alpha s_3^{\beta_3} s_1^{\beta_1} s_2^\gamma s_1^{\delta_1} s_3^{\delta_3} s_2^\varepsilon$. First assume $\gamma = \beta_1$. Then $X = s_2^\alpha s_3^{\beta_3} (s_1^\gamma s_2^\gamma s_1^\gamma) s_3^{\delta_3} s_2^\varepsilon = s_2^\alpha s_3^{\beta_3} s_2^\gamma s_1^\gamma s_2^\gamma s_3^{\delta_3} s_2^\varepsilon$ belongs as before to $A_4^{[2]}$ by lemma 5.1 and elementary transformations, unless $\varepsilon = \gamma$, $\alpha = \gamma$, and then $\beta_3 = -\gamma$. In that case $X = s_2^\alpha s_3^{-\alpha} (s_2^\alpha s_1^\alpha s_2^\alpha) s_3^{-\alpha} s_2^\alpha = s_2^\alpha s_3^{-\alpha} s_1^\alpha s_2^\alpha s_1^\alpha s_3^{-\alpha} s_2^\alpha$ belongs to the RHS. We can thus assume $\gamma \neq \beta_1$ and, applying $\text{Ad } \Delta$, $\gamma \neq \beta_3$, hence we can assume $\beta_1 = \beta_3 = -\gamma$. Then $X = s_2^\alpha s_3^\gamma s_1^\gamma s_2^{-\gamma} s_1^\gamma s_3^\gamma s_2^\varepsilon$, which belongs to the RHS, and this concludes the proof. \square

Lemma 5.3. Let $\alpha, \beta, \varepsilon \in \{-1, 1\}$. Then $s_2^\alpha (s_1 s_3)^\beta s_2^{-\beta} (s_1 s_3)^\beta s_2^\varepsilon$ belongs to

$$A_4^{[2]} + \sum_{\delta \in \{-1, 1\}} Bs_2^\delta (s_1 s_3)^\delta s_2^{-\delta} (s_1 s_3)^\delta s_2^\delta B + \sum_{\delta \in \{-1, 1\}} Bs_2^{-\delta} (s_1 s_3)^\delta s_2^{-\delta} (s_1 s_3)^\delta s_2^{-\delta} B$$

Proof. First assume $\alpha = \beta$. Then $X = s_2^\beta s_1^\beta s_3^\beta s_2^{-\beta} s_3^\beta (s_1^\beta s_2^\varepsilon s_1^{-\beta}) s_1^\beta = s_2^\beta s_1^\beta (s_3^\beta s_2^{-\beta} s_3^\beta s_2^{-\beta}) s_1^\beta s_2^\beta s_1^\beta \in s_2^\beta s_1^\beta s_2^{-\beta} s_3^\beta s_2^{-\beta} s_3^\beta s_1^\beta s_2^\beta + s_2^\beta s_1^\beta u_2 u_3 s_1^\varepsilon s_2^\beta s_1^\beta + s_2^\beta s_1^\beta u_3 u_2 s_1^\varepsilon s_2^\beta s_1^\beta$ by lemma 3.6. Now $s_2^\beta s_1^\beta u_2 u_3 s_1^\varepsilon s_2^\beta s_1^\beta \subset A_4^{[2]}$ and $s_2^\beta s_1^\beta u_3 u_2 s_1^\varepsilon s_2^\beta s_1^\beta \subset A_4^{[2]}$ by lemma 5.1. We thus only need to consider

$$X = (s_2^\beta s_1^\beta s_2^{-\beta}) s_3^\beta s_2^{-\beta} s_3^\beta s_1^\beta s_2^\beta = s_1^{-\beta} s_2^\beta s_1^\beta s_3^\beta s_2^{-\beta} s_3^\beta s_1^\beta s_2^\beta$$

hence, if $\varepsilon = -\beta$, we get

$$X = s_1^{-\beta} s_2^\beta s_3^\beta (s_1^\beta s_2^{-\beta} s_1^{-\beta}) s_3^\beta s_2^\beta = s_1^{-\beta} s_2^\beta s_3^\beta s_2^{-\beta} s_1^{-\beta} (s_2^\beta s_3^\beta s_2^\beta) = s_1^{-\beta} s_2^\beta s_3^\beta s_2^{-\beta} s_1^{-\beta} s_3^\beta s_2^\beta s_3^\beta \in A_4^{[2]}$$

by lemma 5.1. We can thus assume $\varepsilon = \beta$, in which case X clearly belongs to the space we want.

This concludes the case $\alpha = \beta$, hence also the case $\varepsilon = \beta$ by application of Φ and Ψ . Thus we can assume $\alpha = -\beta = \varepsilon$, and the conclusion is clear in this case. \square

Lemma 5.4. For $\alpha, \beta \in \{-1, 1\}$, we have

$$s_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha \in A_4^{[2]} + \sum_{\delta \in \{-1, 1\}} B s_2^\delta (s_1 s_3)^\delta s_2^{-\delta} (s_1 s_3)^\delta s_2^\delta B$$

Proof. The RHS is stable under $\text{Ad } \Delta$ and Φ , hence we can assume $\alpha = \beta = 1$, and thus we only need to consider $X = s_2 s_1 s_3^{-1} s_2 s_1 s_3^{-1} s_2 = s_2 s_1 (s_3^{-1} s_2 s_3^{-1}) s_1 s_2 \in s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2 + s_2 s_1 u_2 u_3 u_2 s_1 s_2$ by lemmas 2.4 and 2.3. We have

$$s_2 s_1 u_2 u_3 u_2 s_1 s_2 \subset \sum_{a \in \{0, 1, -1\}} s_2 s_1 s_2^a u_3 u_2 s_1 s_2$$

and,

- if $a = 0$ we have $s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1 ;
- if $a = 1$ we have $(s_2 s_1 s_2) u_3 u_2 s_1 s_2 = s_1 s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1 ;
- if $a = -1$ we have $(s_2 s_1 s_2^{-1}) u_3 u_2 s_1 s_2 = s_1^{-1} s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1

hence $X \in s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2 + A_4^{[2]}$. The module $s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2$ is R -spanned by the $Y(a) = s_2 s_1 s_2^a s_3 s_2^{-1} s_3 s_1 s_2$ for $a \in \{-1, 0, 1\}$. We have $Y(0) = s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS$, $Y(1) = (s_2 s_1 s_2) s_3 s_2^{-1} s_3 s_1 s_2 = s_1 s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS$ and

$$Y(-1) = (s_2 s_1 s_2^{-1}) s_3 s_2^{-1} s_3 s_1 s_2 = s_1^{-1} s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS,$$

and this concludes the proof. \square

In the braid group on 4 strands, we have

$$\Delta = (s_1 s_2 s_3)(s_1 s_2) s_1 = (s_1 s_3)(s_2 s_1 s_3 s_2) = (s_2 s_1 s_3 s_2)(s_1 s_3)$$

hence the same equalities hold in A_4 . In the remaining part of this section, we let $s = s_2$, $p = s_1 s_3 = s_3 s_1$, hence $\Delta = spsp = psp$. Note that $\Delta p = p\Delta$, $\Delta s = s\Delta$. It follows that $\Delta^2 = p.sps.\Delta = p.\Delta.sps = p(psp)sps = p^2.sps^2ps$, $\Delta^3 = p^2.sps^2ps.\Delta = p^2.\Delta.sps^2ps = p^3.sps^2ps^2p$, and $\Delta^4 = p^4.sps^2ps^2ps^2ps$.

We thus have $\Delta^2 = p^2.sps^2ps$ hence $p^{-2}\Delta^2 \in R^\times sps^{-1}ps + Rspsp + Rsp^2s$ and we known $sp^2s \in A_4^{[2]}$, $(spsp)s = psp^2 \in A_4^{[2]}$ by lemma 5.1. It follows that

$$\begin{aligned} (*) \quad p^{-2}\Delta^2 &\in R^\times sps^{-1}ps + Rspsp^2 + Rsp^2s \\ p^{-2}\Delta^2 &\in R^\times sps^{-1}ps + A_4^{[2]} \end{aligned}$$

Applying Φ , we have $\Phi(\Delta) = \Phi(s_1 s_2 s_3 s_1 s_2 s_1) = s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1^{-1} = (s_1 s_2 s_1 s_3 s_2 s_1)^{-1} = \Delta^{-1}$, hence, since $\Phi(p) = p^{-1}$,

$$(*) \quad p^{-2}\Delta^2 \in R^\times s^{-1}p^{-1}sp^{-1}s^{-1} + A_4^{[2]}$$

Lemma 5.5.

- (1) $s_2^{-1} p s_2^{-1} p s_2^{-1} s_1^{-1} \in u_1^\times s_2 p^{-1} s_2 p^{-1} s_2 + A_4^{[2]}$
- (2) $s_2^{-1} p s_2^{-1} p s_2^{-1} B \subset B s_2 p^{-1} s_2 p^{-1} s_2 + B s_2^{-1} p s_2^{-1} p s_2^{-1} + A_4^{[2]}$
- (3) $s_2 p^{-1} s_2 p^{-1} s_2 B \subset B s_2 p^{-1} s_2 p^{-1} s_2 + B s_2^{-1} p s_2^{-1} p s_2^{-1} + A_4^{[2]}$

Proof. $X = s_2^{-1}ps_2^{-1}ps_2^{-1}.s_1^{-1} = s_2^{-1}ps_2^{-1}s_3(s_1s_2^{-1}.s_1^{-1}) = s_2^{-1}ps_2^{-1}s_3s_2^{-1}s_1^{-1}s_2 = s_2^{-1}s_1(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2 \in s_2^{-1}s_1s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2 + s_2^{-1}s_1u_2u_3s_1^{-1}s_2 + s_2^{-1}s_1u_3u_2s_1^{-1}s_2$ by lemma 3.6. We have $s_2^{-1}s_1u_2u_3s_1^{-1}s_2 \subset A_4^{[2]}$ and $s_2^{-1}s_1u_3u_2s_1^{-1}s_2 \subset A_4^{[2]}$ by lemma 5.1, hence

$$\begin{aligned} X &\in (s_2^{-1}s_1s_2^{-1})s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ &\subset u_1^\times s_2s_1^{-1}(s_2s_3s_2^{-1}s_3s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]}) \\ &\subset u_1^\times s_2s_1^{-1}s_3^{-1}s_2s_3s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ &\subset u_1^\times s_2s_1^{-1}s_3^{-1}s_2p^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_3s_1^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \end{aligned}$$

We know $s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 \in A_4^{[2]}$ by lemma 5.1, $s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1^{-1}s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1^{-1}s_2) = s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_1^{-1} \in A_4^{[2]}$ by lemma 5.1, and $u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 = u_2s_3u_1s_2^{-1}s_1^{-1}s_3s_2$ is the sum of $u_2s_3s_2^{-1}s_1^{-1}s_3s_2 \subset A_4^{[2]}$ (by lemma 5.1) and of the $u_2s_3s_1^\alpha s_2^{-1}s_1^{-1}s_3s_2$ for $\alpha \in \{-1, 1\}$. Since $u_2s_3(s_1^\alpha s_2^{-1}s_1^{-1})s_3s_2 = u_2s_3s_2^{-1}s_1^{-1}(s_2^\alpha s_3s_2) = u_2s_3s_2^{-1}s_1^{-1}s_3s_2s_3^\alpha \subset A_4^{[2]}$ by lemma 5.1, and this proves (1). To get (2) from (1), we use $s_2^{-1}ps_2^{-1}ps_2^{-1}.s_3^{-1} \in u_3^\times s_2p^{-1}s_2p^{-1}s_2^{-1} + A_4^{[2]}$, that we get from (1) by applying $\text{Ad } \Delta$, and the fact that B is generated as a unital R -algebra by s_1^{-1} and s_3^{-1} . This proves (2), and then (3) follows from (2) by a direct application of Φ . \square

From all this we deduce the following.

Lemma 5.6.

- (1) $A_4^{[3]} = A_4^{[2]} + \sum_{\delta \in \{-1, 1\}} Bs^\delta p^\delta s^{-\delta} p^\delta s^\delta + \sum_{\delta \in \{-1, 1\}} Bs^{-\delta} p^\delta s^{-\delta} p^\delta s^{-\delta}$
- (2) $A_4 = A_4^{[3]}$

Proof. As a consequence of lemmas 5.2 and 5.3, we get

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta \in \{-1, 1\}} Bs^\delta p^\delta s^{-\delta} p^\delta s^\delta B + \sum_{\delta \in \{-1, 1\}} Bs^{-\delta} p^\delta s^{-\delta} p^\delta s^{-\delta} B.$$

We know $s^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta}B \subset A_4^{[2]} + \sum_{\varepsilon \in \{-1, 1\}} Bs^{-\varepsilon}p^\varepsilon s^{-\varepsilon}p^\varepsilon s^{-\varepsilon}$ by lemma 5.5 hence

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta \in \{-1, 1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta} + \sum_{\delta \in \{-1, 1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta B$$

and finally $s^\delta p^\delta s^{-\delta}p^\delta s^\delta \in R^\times p^{-\delta} \Delta^{2\delta} + A_4^{[2]}$ by (*), hence $s^\delta p^\delta s^{-\delta}p^\delta s^\delta B \subset p^{-\delta} \Delta^{2\delta} B + A_4^{[2]} = p^{-\delta} B \Delta^{2\delta} + A_4^{[2]} = B \Delta^{2\delta} + A_4^{[2]} \subset Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta + A_4^{[2]}$ and this concludes the proof of (1). Now $A_4^{[3]}$ is a R -submodule of A_4 which contains 1, which is stable under right-multiplication by s_1 and s_3 by definition. Moreover, in view of (1), we have

$$A_4^{[3]}s_2 \subset A_4^{[2]}s + \sum_{\delta \in \{-1, 1\}} Bs^\delta p^\delta s^{-\delta}p^\delta s^\delta s + \sum_{\delta \in \{-1, 1\}} Bs^{-\delta}p^\delta s^{-\delta}p^\delta s^{-\delta} s \subset A_4^{[3]}$$

hence $A_4^{[3]}$ is also stable under right multiplication by s_2 , hence it is a right-ideal of A_4 containing 1, hence (2). \square

We let $x_\pm = s^\pm p^\pm s^\mp p^\pm s^\pm$ and $y_\pm = s^\pm p^\mp s^\pm p^\mp s^\pm$.

Lemma 5.7.

- (1) $sBsp s \subset A_4^{[2]}$
- (2) $sBs^{-1}ps \subset Rsp s^{-1}psA_4^{[2]}$

Proof. The R -module $sBsp s$ is spanned by $s^2ps \in A_4^{[2]}$, the $ss_i sp s \in A_4^{[2]}$ for $i \in \{1, 3\}$ by lemma 5.1, $s(psp s) = s(spsp) = s^2psp \in A_4^{[2]}$, $s_2s_1(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1s_2s_3(s_2^{-1}s_1s_2) = s_2s_1s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$ by lemma 5.1, the image of this latest one by $\text{Ad } \Delta$, and by

$$s_2s_1^{-1}s_3^{-1}s_2s_3s_1s_2 = s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$$

by lemma 5.1, and this proves (1).

Now $sBs^{-1}ps$ is R -spanned by $sp s^{-1}ps$ and

- the $ss^{-1}ps = ps \in A_4^{[2]}$
- the $ssi s^{-1}ps \in A_4^{[2]}$ for $i \in \{1, 3\}$ by lemma 5.1
- $s_2s_1(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1, 3\}$ by lemma 5.1
- $\Delta \cdot s_2s_1s_3^{-1}s_2^{-1}s_3s_1s_2\Delta^{-1} \in A_4^{[2]}$
- $s_2s_1^{-1}(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1^{-1}s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1, 3\}$ by lemma 5.1

and this proves (2). \square

We want to express Δ^3 in terms of the x_{\pm} and y_{\pm} . We recall that $\Delta^2 \in R^{\times}p^2sps^{-1}ps + Rp^3sps^2 + Rp^2sp^2s$ hence $\Delta^3 \in R^{\times}p^2sps^{-1}ps\Delta + Rp^3sps^2\Delta + Rp^2sp^2s\Delta$. We have

- $sps^{-1}ps\Delta \in Rsp\Delta + Rsp\Delta + Rsp\Delta$ and
 - (1) $sps^{-1}\Delta = sps^{-1}(spsp) = sp^2sp \in A_4^{[2]}$,
 - (2) $sps\Delta = sps(spsp) = sps^2psp \in R^{\times}sps^{-1}psp + Rsp\Delta + Rsp^2sp$, and we have $(spsp)sp = psps^2p \in A_4^{[2]}$, $sp^2sp \in A_4^{[2]}$, hence $sps\Delta \in R^{\times}sps^{-1}psp + A_4^{[2]}$.
 - (3) $sps^{-1}\Delta = sps^{-1}spsp = sp^2sp \in A_4^{[2]}$
 hence $sps^2\Delta \in R^{\times}sps^{-1}psp + A_4^{[2]}$.
- $sp^2s\Delta = sp^2s^2psp \in R^{\times}sp^2s^{-1}psp + Rsp^2spsp + Rsp^2psp$, and $sp^2(spsp) = sp^2psps = sp^3sps$, $sp^2psp \in A_4^{[2]}$.

It follows that

$$\Delta^3 \in R^{\times}p^2sp^2sp^2s + Rp^3sps^{-1}psp + Rp^2sp^2s^{-1}psp + Rp^2sp^3sps + A_4^{[2]}$$

From (*) we have $p^2sps^{-1}ps \in \Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in p\Delta^2p + A_4^{[2]} = p^2\Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in R^{\times}p^4.sps^{-1}ps + A_4^{[2]}$. By lemma 5.7, we have $sp^2s^{-1}ps \in Rx_+ + A_4^{[2]}$ hence $p^2sp^2s^{-1}psp \in Rp^2x_+p + A_4^{[2]} \subset Bx_+ + A_4^{[2]}$. Since $sp^3sps \in A_4^{[2]}$ this leads to

$$\Delta^3 \in R^{\times}p^2sp^2sp^2s + Bx_+ + A_4^{[2]}.$$

Since $s_i^2 = as_i + b + cs_i^{-1}$ we have $p^2 = s_1^2s_3^2 = (as_1 + b + cs_1^{-1})(as_3 + b + cs_3^{-1}) \in a^2p + c^2p^{-1} + W$ with W the R -span of $s_1s_3^{-1}, s_3s_1^{-1}, s_1, s_3, s_1^{-1}, s_3^{-1}, 1$. After easy applications of lemma 5.1 it follows that $sp^2sp^2s \in c^4sp^{-1}sp^{-1}s + RspBs + RsBsp + A_4^{[2]}$. Since $spsBs + sBsp \subset A_4^{[2]}$ by lemma 5.7 we get

$$\Delta^3 \in c^4p^2y_+ + Bx_+ + A_4^{[2]}$$

and

$$\Delta^{-3} = \Phi(\Delta^3) \in c^{-4}p^{-2}y_- + Bx_+ + A_4^{[2]}$$

Now we have $\Delta^3s_1 = s_3\Delta^3 \in c^4s_3p^2y_+ + Bx_+ + A_4^{[2]}$ and $\Delta^3s_1 \in c^4p^2y_+s_1 + Bx_+B + A_4^{[2]}$, $\Delta^3s_1 \in c^4p^2u_1^{\times}y_- + Bx_+B + A_4^{[2]}$ by lemma 5.5 (1), $\Delta^3s_1 \in c^4p^2u_1^{\times}y_- + Bx_+ + A_4^{[2]}$ by using $p^{-2}\Delta^2 \in R^{\times}x_+ + A_4^{[2]}$. It follows that $c^4s_3p^2y_+ \in c^4p^2u_1^{\times}y_- + Bx_+ + A_4^{[2]}$ hence

$$\begin{cases} y_+ & \in By_- + Bx_+ + A_4^{[2]} \\ y_- & \in By_+ + Bx_+ + A_4^{[2]} \end{cases}$$

As a consequence we get the following.

Proposition 5.8.

$$A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_+ = A_4^{[2]} + Bx_+ + Bx_- + By_-$$

For $x \in A_4^{\times}$, we let $[x]$ denote its class in $B^{\times} \setminus A_4^{\times} / B^{\times}$, and we write $x \sim y$ for $[x] = [y]$.

Lemma 5.9. *Let $E_2 = \{s_2^{\alpha}s_1^{\beta}s_3^{\gamma}s_2^{\delta} \mid \alpha, \beta, \gamma, \delta \in \{0, 1, -1\}\} \subset A_4^{\times}$. The image of $[E_2]$ of E_2 in $B^{\times} \setminus A_4^{\times} / B^{\times}$ has cardinality at most 13, and is equal to \mathcal{S}_2 , with*

$$\mathcal{S}_2 = \{[1], [s_2], [s_2^{-1}], [s_2s_1^{-1}s_2], [s_2s_3^{-1}s_2], [s_2s_1s_3s_2], [s_2s_1^{-1}s_3s_2], [s_2s_1^{-1}s_3^{-1}s_2], [s_2s_1s_3^{-1}s_2^{-1}], [s_2s_1^{-1}s_3s_2^{-1}], [s_2s_1s_3^{-1}s_2^{-1}], [s_2^{-1}s_1s_3s_2^{-1}], [s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}]\}$$

Proof. Clearly $\mathcal{S}_2 \subset [E_2]$, hence we only need to prove $[E_2] \subset \mathcal{S}_2$. In view of $s_2^\alpha s_i^\alpha s_2^\alpha = s_i^\alpha s_2^\alpha s_i^\alpha$, $s_2^{-1} s_i^\alpha s_2 = s_i s_2^\alpha s_i^{-1}$, $s_2 s_i^\alpha s_2^{-1} = s_i^{-1} s_2^\alpha s_i$ for $\alpha \in \{-1, 1\}$ and $i \in \{1, 3\}$, we have $[s_2^\alpha s_i^\beta s_2^\gamma] \in \mathcal{S}_2$ for all α, β, γ . Among the $s_2 s_1^\alpha s_3^\beta s_2$ for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2 s_1^\alpha s_3^\beta s_2] \in \mathcal{S}_2$ because $s_2 s_1^\alpha s_3^\beta s_2 \sim s_2 s_1^{-1} s_3 s_2$: indeed, we have the identity $s_2 s_1^{-1} s_3 s_2 = s_1^{-1} s_3 (s_2 s_1 s_3^{-1} s_2) s_1^{-1} s_3$ in the braid group on 4 strands (because $s_1^{-1} s_3 s_2 s_1 s_3^{-1} s_2 s_1^{-1} s_3 = s_1^{-1} (s_3 s_2 s_3^{-1}) s_1 s_2 s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_3 (s_2 s_1 s_2) s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_3 s_1 s_2 s_1 s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_1 (s_3 s_2 s_3) = (s_1^{-1} s_2^{-1} s_1) s_2 s_3 s_2 = s_2 s_1^{-1} s_2^{-1} s_2 s_3 s_2 = s_2 s_1^{-1} s_3 s_2$). Among the $s_2 s_1^\alpha s_3^\beta s_2^{-1}$ for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2 s_1^\alpha s_3^\beta s_2^{-1}] \in \mathcal{S}_2$ because $s_2 s_1 s_3 s_2^{-1} \sim s_2 s_1 s_3^{-1} s_2$: indeed, we have $s_1 (s_2 s_1 s_3 s_2^{-1}) s_3^{-1} = (s_1 s_2 s_1) s_3 s_2^{-1} s_3^{-1} = s_2 s_1 (s_2 s_3 s_2^{-1}) s_3^{-1} = s_2 s_1 s_3^{-1} s_2 s_3 s_3^{-1} s_2 = s_2 s_1 s_3^{-1} s_2$.

Again for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2^{-1} s_1^\alpha s_3^\beta s_2] \in \mathcal{S}_2$ because of the following identities

- (1) $s_2^{-1} s_1^{-1} s_3 s_2 \sim s_2 s_1 s_3^{-1} s_2^{-1}$
- (2) $s_2^{-1} s_1 s_3 s_2 \sim s_2 s_1 s_3^{-1} s_2$
- (3) $s_2^{-1} s_1 s_3^{-1} s_2 \sim s_2 s_1^{-1} s_3 s_2^{-1}$
- (4) $s_2^{-1} s_1^{-1} s_3^{-1} s_2 \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$

We prove these identities now. We have $s_3 (s_2 s_3^{-1} s_1 s_2^{-1}) s_1^{-1} = (s_3 s_2 s_3^{-1}) s_1 s_2^{-1} s_1^{-1} = s_2^{-1} s_3 (s_2 s_1 s_2^{-1}) s_1^{-1} = s_2^{-1} s_3 s_1^{-1} s_2 s_1 s_1^{-1} = s_2^{-1} s_3 s_1^{-1} s_2$ hence $s_2 s_3^{-1} s_1 s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2$ that is (1). By applying $\text{Ad } \Delta$ this implies $s_2 s_1^{-1} s_3 s_2^{-1} \sim s_2^{-1} s_1 s_3^{-1} s_2$ that is (3). We have $s_3^{-1} (s_2^{-1} s_1 s_3 s_2) s_1 = (s_3^{-1} s_2^{-1} s_3) s_1 s_2 s_1 = s_2 s_3^{-1} (s_2^{-1} s_1 s_2) s_1 = s_2 s_3^{-1} s_1 s_2 s_1^{-1} s_1 = s_2 s_3^{-1} s_1 s_2$ hence $s_2^{-1} s_1 s_3 s_2 \sim s_2 s_3^{-1} s_1 s_2$ that is (2).

We have $s_1 (s_2^{-1} s_1^{-1} s_3 s_2^{-1}) s_3^{-1} = (s_1 s_2^{-1} s_1^{-1}) s_3 s_2^{-1} s_3^{-1} = s_2^{-1} s_1^{-1} (s_2 s_3 s_2^{-1}) s_3^{-1} = s_2^{-1} s_1^{-1} s_3^{-1} s_2 s_3 s_3^{-1} = s_2^{-1} s_1^{-1} s_3^{-1} s_2$ hence $s_2^{-1} s_1^{-1} s_3 s_2^{-1} \sim s_2^{-1} s_1^{-1} s_3^{-1} s_2$. Moreover, we have $s_1 (s_2 s_1^{-1} s_3^{-1} s_2^{-1}) s_3^{-1} = s_1 s_2 s_1^{-1} (s_3^{-1} s_2^{-1} s_3) = s_1 (s_2 s_1^{-1} s_2^{-1}) s_3^{-1} s_2^{-1} = s_1 s_1^{-1} s_2^{-1} s_1 s_3^{-1} s_2^{-1} = s_2^{-1} s_1 s_3^{-1} s_2^{-1}$ hence $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_1 s_3^{-1} s_2^{-1}$. Applying Δ we get $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2^{-1}$, hence $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2^{-1} \sim s_2^{-1} s_1^{-1} s_3^{-1} s_2$ hence (4).

Now, for $\alpha, \beta \in \{-1, 1\}$, we have $[s_2^{-1} s_1^\alpha s_3^\beta s_2^{-1}] \in \mathcal{S}_2$ because $s_2^{-1} s_1 s_3^{-1} s_2^{-1} \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$ and $s_2^{-1} s_1^{-1} s_3 s_2^{-1} \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$ as we proved above, and this concludes the proof. \square

From this we get

$$\begin{aligned} A_4 &= \sum_{\sigma \in [E_2]} B \sigma B + B x_+ + B x_- + B y_- \\ &= \sum_{\sigma \in \mathcal{S}_2} B \sigma B + B x_+ + B x_- + B y_- \end{aligned}$$

We write $\mathcal{S}_2 = \mathcal{S}_2^1 \cup \mathcal{S}_2^\Delta \cup \mathcal{S}_2^\alpha \cup \mathcal{S}_2^\beta \cup \mathcal{S}_2^0$ with

$$\begin{aligned} \mathcal{S}_2^1 &= \{[1], [s_2], [s_2^{-1}]\} \\ \mathcal{S}_2^\Delta &= \{[s_2 s_1 s_3 s_2], [s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}]\} \\ \mathcal{S}_2^\alpha &= \{[s_2 s_1^{-1} s_2], [s_2 s_3^{-1} s_2]\} \\ \mathcal{S}_2^\beta &= \{[s_2 s_1 s_3^{-1} s_2^{-1}], [s_2 s_1^{-1} s_3 s_2^{-1}]\} \\ \mathcal{S}_2^0 &= \{[s_2 s_1^{-1} s_3 s_2], [s_2 s_1^{-1} s_3^{-1} s_2], [s_2 s_1^{-1} s_3^{-1} s_2^{-1}], [s_2^{-1} s_1 s_3 s_2^{-1}]\} \end{aligned}$$

Recall that $B = u_1 u_3 = u_3 u_1$, with u_i the unital subalgebra generated by s_i . We prove the following.

Lemma 5.10.

- (1) $s_2 s_1 s_3 s_2 B \subset B s_2 s_1 s_3 s_2$, $s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1} B \subset B s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}$
- (2) $s_2 s_1^{-1} s_2 B \subset B s_2 s_1^{-1} s_2 u_3 + A_4^{[1]}$, $s_2 s_3^{-1} s_2 B \subset B s_2 s_3^{-1} s_2 u_1 + A_4^{[1]}$
- (3) $s_2 s_1 s_3^{-1} s_2^{-1} B \subset B s_2 s_1 s_3^{-1} s_2^{-1} u_1$, $s_2 s_1^{-1} s_3 s_2^{-1} B \subset B s_2 s_1^{-1} s_3 s_2^{-1} u_3$

Proof. We have $\Delta = s_1 s_3 (s_2 s_1 s_3 s_2) = (s_2 s_1 s_3 s_2) s_1 s_3$ and $\Delta B = B \Delta$ hence $s_2 s_1 s_3 s_2 B = (s_1 s_3)^{-1} \Delta B = B (s_1 s_3)^{-1} \Delta = B (s_2 s_1 s_3 s_2)$. Applying Φ (or considering Δ^{-1}) we get $s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1} B = B s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}$ hence (1).

By lemma 3.6 we have $(s_2 s_1^{-1} s_2) s_1^{-1} \in s_1^{-1} (s_2 s_1^{-1} s_2) + A_4^{[1]}$ hence $(s_2 s_1^{-1} s_2) u_1 \subset u_1 (s_2 s_1^{-1} s_2) + A_4^{[1]}$ and $(s_2 s_1^{-1} s_2) u_1 \subset B (s_2 s_1^{-1} s_2) + A_4^{[1]}$. Since $B = u_1 u_3$ this yields $(s_2 s_1^{-1} s_2) B = (s_2 s_1^{-1} s_2) u_1 u_3 \subset B (s_2 s_1^{-1} s_2) u_3 + A_4^{[1]}$. Using $\text{Ad } \Delta$ this implies $(s_2 s_3^{-1} s_2) B \subset B (s_2 s_3^{-1} s_2) u_1 + A_4^{[1]}$, hence (2). Finally, $(s_2 s_1 s_3^{-1} s_2^{-1}) s_3 = s_2 s_1 (s_3^{-1} s_2^{-1} s_3) = (s_2 s_1 s_2) s_3^{-1} s_2^{-1} = s_1 (s_2 s_1 s_3^{-1} s_2^{-1})$ hence $(s_2 s_1 s_3^{-1} s_2^{-1}) u_3 \subset$

$B(s_2s_1s_3^{-1}s_2^{-1})$ whence, using $B = u_3u_1$, $(s_2s_1s_3^{-1}s_2^{-1})B \subset B(s_2s_1s_3^{-1}s_2^{-1})u_1$ and, applying $\text{Ad } \Delta$, $(s_2s_1^{-1}s_3s_2^{-1})B \subset B(s_2s_1^{-1}s_3s_2^{-1})u_3$, which proves (3). \square

Proposition 5.11.

(1) $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$ is equal to

$$B + \sum_{a,b \in \{0,1,-1\}} Bs_2s_1^a s_3^b + \sum_{a,b \in \{0,1,-1\}} Bs_2^{-1}s_1^a s_3^b$$

(2) $A_4^{[2]} = Bu_2A_4^{[1]} = A_4^{[1]}u_2B$ is equal to

$$\begin{aligned} A_4^{[1]} + \sum_{x \in \mathcal{S}_2^\Delta} Bx + \sum_{a \in \{0,1,-1\}} Bs_2s_1^{-1}s_2s_3^a + \sum_{a \in \{0,1,-1\}} Bs_2s_3^{-1}s_2s_1^a + \sum_{a \in \{0,1,-1\}} Bs_2s_1s_3^{-1}s_2^{-1}s_1^a \\ + \sum_{a \in \{0,1,-1\}} Bs_2s_1^{-1}s_3s_2^{-1}s_3^a + \sum_{\substack{x \in \mathcal{S}_2^0 \\ a,b \in \{0,1,-1\}}} Bxs_1^a s_3^b \end{aligned}$$

(3) $A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_-$

Proof. (1) is clear, (3) has been proved before, and (2) is an immediate consequence of $A_4^{[2]} = A_4^{[1]} + \sum_{x \in \mathcal{S}_2} BxB$ and of lemma 5.10. \square

Corollary 5.12. *As a B -module, A_4 is generated by 72 elements, which are images of elements of the braid group on 4 strands.*

Proof. By proposition 5.11, $A_4^{[1]}$ is generated by $1 + 9 + 9 = 19$ elements, $A_4^{[2]}$ by $A_4^{[1]}$ and $|\mathcal{S}_2^\Delta| + 4 \times 3 + 9 \times |\mathcal{S}_2^0| \leq 2 + 12 + 9 \times 4 = 50$ elements, and $A_4^{[3]}$ by $A_4^{[2]}$ and 3 elements. Thus $A_4 = A_4^{[3]}$ is generated by 72 elements, all originating from the braid group. \square

6. THE ALGEBRA A_5

Recall $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} \in A_4$. Our first goal in this section is to prove the following theorem.

Theorem 6.1.

$$\begin{aligned} A_5 = & A_4 + A_4s_4A_4 + A_4s_4^{-1}A_4 + A_4s_4s_3^{-1}s_4A_4 + A_4s_4^{-1}s_3s_2^{-1}s_3s_4^{-1}A_4 + A_4s_4s_3^{-1}s_2s_3^{-1}s_4A_4 \\ & + A_4s_4^{-1}w^+s_4^{-1}A_4 + A_4s_4w^-s_4A_4 + A_4s_4^{-1}w^-s_4^{-1}A_4 + A_4s_4w^+s_4A_4 + A_4s_4w^-s_4w^-s_4A_4 \\ & + A_4s_4w^+s_4^{-1}w^+s_4A_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4 \end{aligned}$$

We denote again U the right-hand side. We let $A_5^{(0)} = A_4$ and $A_5^{(n+1)} = A_5^{(n)}u_4A_4$. This defines an increasing sequence of A_4 sub-bimodules of A_5 . An immediate consequence of theorem 4.1 is $sh(A_4) \subset U$. Also, we have $u_4 \subset U$ hence $A_5^{(1)} = A_4u_4A_4 \subset U$.

Lemma 6.2. $u_4A_4u_4 \subset U$, hence $A_5^{(2)} = A_4u_4A_4u_4A_4 \subset U$.

Proof. According to theorem 4.1, we have $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w^- + A_3w^+$, hence $u_4A_4u_4 \subset A_3u_4 + A_4u_4u_3u_4A_4 + A_4u_4s_3s_2^{-1}s_3u_4A_3 + A_3u_4w^-u_4 + A_3u_4w^+u_4$. We have $A_3u_4 + A_4u_4u_3u_4A_4 + A_4u_4s_3s_2^{-1}s_3u_4A_3 \subset A_4sh(A_4)A_4 \subset A_4UA_4 \subset U$. Moreover, since by definition $s_4^\alpha w^\beta s_4^\alpha \in U$ for all $\alpha, \beta \in \{-1, 1\}$, we have $s_4A_4s_4 \subset U$, $s_4^{-1}A_4s_4^{-1} \subset U$, and we only need to prove $s_4w^\pm s_4^{-1} \in U$ and $s_4^{-1}w^\pm s_4 \in U$. We have $w^\pm \in s_3^\alpha A_3s_3^\alpha$ for some $\alpha \in \{-1, 1\}$, hence $s_4^\alpha w^\pm s_4^{-\alpha} \in s_4^\alpha s_3^\alpha A_3s_3^\alpha s_4^{-\alpha} = s_3^{-\alpha}(s_3^\alpha s_4^\alpha s_3^\alpha)A_3s_3^\alpha s_4^{-\alpha} = s_3^{-\alpha}s_4^\alpha s_3^\alpha s_4^\alpha A_3s_3^\alpha s_4^{-\alpha} \subset A_4s_4^\alpha s_3^\alpha A_3(s_4^\alpha s_3^\alpha s_4^{-\alpha}) \subset A_4s_4^\alpha s_3^\alpha A_3s_3^{-\alpha} s_4^\alpha s_3^\alpha$ by lemma 2.1. Now

$$A_4s_4^\alpha s_3^\alpha A_3s_3^{-\alpha} s_4^\alpha s_3^\alpha \subset A_4(s_4^\alpha A_4s_4^\alpha)A_4 \subset A_4UA_4 \subset U,$$

as we already proved. \square

6.1. The A_4 -bimodule $A_5^{(3)}/A_5^{(2)}$: first reduction.

Proposition 6.3. *If $p \leq 5$, $q \leq 5$ and $(p, q) \neq (5, 5)$, then for all $x \in u_4 u_{i_1} \dots u_{i_p} u_4 u_{j_1} \dots u_{j_q} u_4$ we have $x \in A_4 u_4 A_4 u_4 A_4$, for all choices of $i_1, \dots, i_p, j_1, \dots, j_q \in \{1, 2, 3\}$, unless $(p, q) \in \{(5, 4), (4, 5)\}$ and $x \in s_4 u_3 u_2 u_1 u_3 u_2 s_4 u_1 u_3 u_2 u_3 s_4 \cup s_4^{-1} u_3 u_2 u_1 u_3 u_2 s_4^{-1} u_1 u_3 u_2 u_3 s_4^{-1}$.*

Proof. Note that $sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. By application of Ψ we may assume $p \geq q \geq 1$. We prove the statement by induction on (p, q) , using lexicographic ordering. By commutation relations we can assume $i_1 \notin \{1, 2\}$ hence $i_1 = 3$, and similarly $j_q = 3$. In case $(p, q) = (1, 1)$ we have then $u_4 u_3 u_4 u_3 u_4 \subset sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. More generally, in the cases $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, using only commutation relations we check that the corresponding algebras are necessarily included in $A_4 sh(A_4) A_4 \subset A_4 u_4 A_4 u_4 A_4$.

If $(p, q) = (3, 2)$, the only case which is not clearly included in $A_4 sh(A_4) A_4$ is $u_4 u_3 u_2 u_1 u_4 u_2 u_3 u_4 = u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4$, and we have $u_4 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 + u_3 u_4 u_3$ by theorem 3.2 hence $u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 u_2 u_1 u_2 u_3 u_4 + u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4 u_3 u_4 + A_4 u_4 A_4 u_4 A_4$. Again $s_4 u_3 u_4 \subset s_4^{-1} s_3 s_4^{-1} u_3 + u_3 u_4 u_3$ by theorem 3.2 hence $u_3 s_4 s_3^{-1} u_2 u_1 u_2 (s_4 u_3 u_4) \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} + A_4 u_4 A_4 u_4 A_4$ and $u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} = u_3 (s_4 s_3^{-1} s_4^{-1}) u_2 u_1 u_2 s_3 s_4^{-1} = u_3 s_3^{-1} s_4^{-1} s_3 u_2 u_1 u_2 s_3 s_4^{-1} \subset A_4 u_4 A_4 u_4 A_4$.

If $(p, q) = (3, 3)$, the corresponding algebra is either included in $A_4 sh(A_4) A_4 \subset U$, or can be reduced using commutation relations to the case $(3, 2)$, or we are dealing with the remaining case $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4$ (or its image under the natural anti-isomorphism $u_4 u_3 u_2 u_3 u_4 u_1 u_2 u_3 u_4$). We want to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$. Up to using the natural isomorphism induced by $s_i \mapsto s_i^{-1}$, we only need to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 \subset A_4 u_4 A_4 u_4 A_4$. We have $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 = u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4$ and we know from theorem 3.2 that $u_4 u_3 u_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} + u_3 u_4 u_3$, hence $u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 + A_4 u_4 A_4 u_4 A_4$. Now, using $u_3 u_2 u_3 \subset s_3 s_2^{-1} s_3 u_2 + u_2 u_3 u_3$ we have

$$s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 u_2 s_4 + s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4.$$

But $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4 = s_4^{-1} s_3 u_2 u_1 s_4^{-1} u_2 u_3 s_4 u_2 \subset A_4 u_4 A_4 u_4 A_4$ by the induction assumption, hence $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 s_4 u_2 + A_4 u_4 A_4 u_4 A_4$. Now we need to prove $s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1^{-1} s_3 s_2^{-1} s_3 s_4 \in A_4 u_4 A_4 u_4 A_4$ for $\alpha, \beta \in \{-1, 1\}$. If $\alpha = 1$, this holds true because

$$\begin{aligned} s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} s_3 s_4 &= s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} (s_3 s_4 s_3) s_3^{-1} \\ &= s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} s_4 s_3 s_4 s_3^{-1} \\ &= s_4^{-1} s_3 s_2 u_1 (s_4^{-1} s_3 s_4) s_2^{-1} s_3 s_4 s_3^{-1} \\ &= s_4^{-1} s_3 s_2 u_1 s_3 s_4 s_3^{-1} s_2^{-1} s_3 s_4 s_3^{-1} \\ &= s_4^{-1} (s_3 s_2 s_3) u_1 s_4 (s_3^{-1} s_2^{-1} s_3) s_4 s_3^{-1} \\ &= s_4^{-1} s_2 s_3 s_2 u_1 s_4 s_2 s_3^{-1} s_2^{-1} s_4 s_3^{-1} \\ &= s_2 s_4^{-1} s_3 s_2 u_1 s_4 s_2 s_3^{-1} s_4 s_2^{-1} s_3^{-1} \\ &\subset A_4 u_4 A_4 u_4 A_4 \end{aligned}$$

by the induction assumption. We thus assume $\alpha = -1$. If $\beta = 1$, then

$$\begin{aligned} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 (s_3 s_2^{-1} s_3) s_4 &\subset s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \text{ (lemmas 2.4 + 2.3)} \\ &\subset s_1 s_1^{-1} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \\ &\subset s_1 s_4^{-1} s_3 s_4^{-1} (s_1^{-1} s_2^{-1} s_1) s_3^{-1} s_2 s_3^{-1} s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \\ &\subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_2 s_3^{-1} s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \\ &\subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} sh(A_3) s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \\ &\subset s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 u_2 s_4 u_2 &+ A_4 u_4 A_4 u_4 A_4 \end{aligned}$$

by theorem 3.2 and the induction assumption, and we already proved $s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 u_2 s_4 = s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 s_4 u_2 \subset A_4 u_4 A_4 u_4 A_4$. The remaining case is then $(\alpha, \beta) = (-1, -1)$, for which we have

$$\begin{aligned}
s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_4 &\subset s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}s_3^{-1}s_2s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \text{ (lemma 2.4)} \\
&\subset s_3s_3^{-1}(s_4^{-1}s_3s_4^{-1})s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \\
&\subset s_3(s_4^{-1}s_3s_4^{-1})s_3^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \text{ (lemma 2.3)} \\
&\subset A_4s_4^{-1}s_3s_4^{-1}(s_3^{-1}s_2^{-1}s_3^{-1})s_1^{-1}s_2s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1^{-1}s_2)s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1(s_3^{-1}s_2^{-1}s_3^{-1})s_4s_1^{-1}A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2^{-1}s_4s_1^{-1}A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_4s_2^{-1}A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4s_4^{-1}s_3s_2^{-1}s_1s_4^{-1}s_2^{-1}s_3^{-1}s_4A_4 && +A_4u_4A_4u_4A_4 \\
&\subset A_4u_4A_4u_4A_4 && \text{(induction assumption)}
\end{aligned}$$

and this concludes the case $(p, q) = (3, 3)$.

All cases $(4, q)$ for $q = 1, 2, 4$ can be easily reduced to smaller cases by using commutation relations and relations $u_i u_j u_i u_j = u_j u_i u_j u_i$. Most cases for $(4, 3)$ can also be reduced this way, except for one remaining case $u_4 u_3 u_2 u_3 u_1 u_4 u_3 u_2 u_3 u_4$. Using Φ , we only need to prove $u_4 u_3 u_2 u_3 u_1 s_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$. Using the induction assumption and theorem 3.2 on $sh(A_3)$, we get

$$\begin{aligned}
u_4(u_3 u_2 u_3) u_1 s_4 (u_3 u_2 u_3) u_4 &\subset u_4 u_2 s_3 s_2^{-1} s_3 u_1 s_4 s_3 s_2^{-1} s_3 u_2 u_4 && +A_4 u_4 A_4 u_4 A_4 \\
&\subset A_3 u_4 s_3 s_2^{-1} u_1 (s_3 s_4 s_3) s_2^{-1} s_3 u_4 A_3 && +A_4 u_4 A_4 u_4 A_4 \\
&\subset A_3 u_4 s_3 s_2^{-1} u_1 s_4 s_3 s_4 s_2^{-1} s_3 u_4 A_3 && +A_4 u_4 A_4 u_4 A_4 \\
&\subset A_3 (u_4 s_3 s_4) s_2^{-1} u_1 s_3 s_2^{-1} (s_4 s_3 u_4) A_3 && +A_4 u_4 A_4 u_4 A_4 \\
&\subset A_3 u_3 u_4 u_3 s_2^{-1} u_1 s_3 s_2^{-1} u_3 u_4 u_3 A_3 && +A_4 u_4 A_4 u_4 A_4
\end{aligned}$$

by lemma 2.1, which proves the claim.

We now deal with the cases $(5, q)$ with $1 \leq q < 5$. We can assume that $u_{i_1} \dots u_{i_p} = u_3 u_2 u_1 u_2 u_3$ or $u_{i_1} \dots u_{i_p} = u_3 u_2 u_1 u_3 u_2$, because otherwise we can reduce to smaller cases by using commutation relations and the relation $u_a u_b u_a u_b = u_b u_a u_b u_a$. From this remark one easily checks that the cases $(5, 1)$ are readily reduced to smaller cases, and also the cases $(5, 2)$ except for the case $u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_3 u_4 = u_4 u_3 u_2 u_1 u_2 u_3 u_2 u_4 u_3 u_4$ that we tackle now : we have $u_3 u_2 u_1 u_2 u_3 u_2 \subset A_4 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3 + A_3 u_3 u_2 u_1 u_2 u_3$ by theorem 4.1, hence

$$\begin{aligned}
u_4 u_3 u_2 u_1 u_2 u_3 u_2 u_4 u_3 u_4 &\subset u_4 A_3 u_3 A_3 u_4 u_3 u_4 + u_4 A_3 u_3 u_2 u_3 A_3 u_4 u_3 u_4 + u_4 A_3 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_4 \\
&\subset A_3 u_4 u_3 u_4 A_3 u_3 u_4 + A_3 u_4 u_3 u_2 u_3 u_4 A_3 u_3 u_4 + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_4 \\
&\subset A_3 u_4 u_3 u_4 u_2 u_1 u_2 u_1 u_3 u_4 + A_3 u_4 u_3 u_2 u_3 u_4 u_2 u_1 u_2 u_1 u_3 u_4 \\
&\quad + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_4 \\
&\subset A_3 u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_2 + A_3 u_4 u_3 u_2 u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_2 \\
&\quad + A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_4
\end{aligned}$$

using $A_3 = u_2 u_1 u_2 u_1$, and we are thus reduced to smaller cases.

When $(p, q) = (5, 3)$, the only nontrivial case (up to commutation and $u_a u_b u_a u_b = u_b u_a u_b u_a$ relations) is $u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_3 u_4$. We have $u_2 u_3 u_4 u_3 u_2 u_3 u_4 \subset sh(A_4) \subset A_4 u_4 A_4 + sh(A_3) u_4 u_3 u_4 A_4 + u_4 u_3 u_2 u_3 u_4 A_4$ by theorem 4.1, hence

$$u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4 + u_4 u_3 u_2 u_1 sh(A_3) u_4 u_3 u_4 A_4 + u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4 A_4$$

and we have $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$ by the induction assumption, and, since $sh(A_3) = u_2 u_3 u_2 u_3$ by theorem 3.2,

$$\begin{aligned}
u_4 u_3 u_2 u_1 sh(A_3) u_4 u_3 u_4 &\subset u_4 u_3 u_2 u_1 u_2 u_3 u_2 (u_3 u_4 u_3 u_4) &= u_4 u_3 u_2 u_1 u_2 u_3 u_2 u_4 u_3 u_4 u_3 \\
& &= u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_3 u_4 u_3
\end{aligned}$$

and we are reduced to case $(5, 2)$.

When $(p, q) = (5, 4)$, the only nontrivial cases are

$$u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_2 u_3 u_4 \text{ and } u_4 u_3 u_2 u_1 u_3 u_2 u_4 u_3 u_1 u_2 u_3 u_4.$$

In the first case, $u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4 = u_4u_3u_2u_1u_2u_3u_2u_1u_2u_4u_3u_4 \subset u_4A_4u_4u_3u_4$. By theorem 4.1, we have $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ hence

$$\begin{aligned} u_4A_4u_4u_3u_4 &\subset u_4A_3u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_4 \\ &\subset A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4 \\ &\subset A_4u_4A_4u_4A_4 \end{aligned}$$

by the induction assumption and $A_3 = u_2u_1u_2u_1$.

In the second case, we need to consider the sets $s_4^\alpha u_3u_2u_1u_3u_2s_4^\beta u_3u_1u_2u_3s_4^\gamma$ with $\alpha, \beta, \gamma \in \{-1, 1\}$, and we can assume that two of them have distinct signs, otherwise we are in the exceptional case of the statement. Up to using Φ and Ψ , we can assume $\gamma = 1$ and $\beta = -1$. We are thus considering expressions of the type $u_4u_3u_2u_1u_3u_2s_4^{-1}u_3u_1u_2u_3s_4 = u_4u_3u_2u_3u_1u_2u_1s_4^{-1}u_3u_2u_3s_4$. Notice that

$$\begin{aligned} u_4u_3u_2u_3(u_2u_1u_2)s_4^{-1}u_3u_2u_3s_4 &= u_4(u_3u_2u_3u_2)u_1u_2s_4^{-1}u_3u_2u_3s_4 \\ &= u_4u_2u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 = u_2u_4u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 \end{aligned}$$

hence reduces to smaller cases. As a consequence, among the natural spanning set of $u_1u_2u_1$, only the $s_1^\alpha s_2^{-\alpha} s_1^\alpha$ do not reduce to smaller cases, and so we may restrict ourselves to these. Moreover, using $u_3u_2u_3 \subset u_2s_3^{-1}s_2s_3^{-1} + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3s_2^{-1}s_3u_2 + u_2u_3u_2$ we are reduced to expressions of the form $u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_4^{-1}s_3s_2^{-1}s_3s_4$. We then have

$$\begin{aligned} u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_4^{-1}s_3s_2^{-1}s_3s_4 &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_4^{-1}s_3s_2^{-1}(s_3s_4s_3)s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_4^{-1}s_3s_2^{-1}s_4s_3s_4s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha (s_4^{-1}s_3s_4)s_2^{-1}s_3s_4s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_3s_4s_3^{-1}s_2^{-1}s_3s_4s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_3s_4(s_3^{-1}s_2^{-1}s_3)s_4s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_3s_4s_2s_3^{-1}s_2^{-1}s_4s_3^{-1} \\ &= u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_3s_4s_2s_3^{-1}s_4s_2^{-1}s_3^{-1} \end{aligned}$$

so we now need to prove that $u_4s_3^{-1}s_2s_3^{-1}s_1^\alpha s_2^{-\alpha} s_1^\alpha s_3s_4s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$. When $\alpha = 1$ we get

$$\begin{aligned} u_4s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_1s_3s_4s_2s_3^{-1}s_4 &= u_4s_3^{-1}s_2s_1(s_3^{-1}s_2^{-1}s_3)s_1s_4s_2s_3^{-1}s_4 \\ &= u_4s_3^{-1}(s_2s_1s_2)s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ &= u_4s_3^{-1}s_1s_2s_1s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ &= s_1u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ &= s_1u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}s_4s_1s_2s_3^{-1}s_4 \\ &\subset A_4u_4A_4u_4A_4 \end{aligned}$$

by the induction assumption. When $\alpha = -1$ we get

$$\begin{aligned} u_4s_3^{-1}s_2s_3^{-1}s_1^{-1}s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 &= u_4s_3^{-1}s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1^{-1}s_4s_2s_3^{-1}s_4 \\ &= u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_2^{-1}s_1^{-1}s_4s_2s_3^{-1}s_4 \\ &= u_4s_3^{-1}s_2s_1^{-1}s_2s_3(s_2^{-1}s_1^{-1}s_2)s_4s_3^{-1}s_4 \\ &= u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_1^{-1}s_4s_3^{-1}s_4 \\ &= u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_4s_3^{-1}s_4s_1^{-1} \\ &= u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_1s_2^{-1}s_3^{-1}s_4s_1^{-1} \\ &\subset A_4u_4A_4u_4A_4 \end{aligned}$$

by the induction assumption.

This concludes the case (5, 4) and the proof of the proposition. \square

Lemma 6.4.

$$u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)u_2 + A_4u_4A_4u_4A_4$$

Proof. By proposition 6.3 it is enough to prove

$$s_4u_3u_2u_3u_1u_2u_1s_4u_3u_2u_3s_4 \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4u_2 + A_4u_4A_4u_4A_4$$

and, as noted in the proof of proposition 6.3, we can restrict to the forms $s_4u_3u_2u_3s_1^\alpha s_2^{-\alpha} s_1^\alpha s_4u_3u_2u_3s_4$. Moreover, since $s_1s_2^{-1}s_1 = s_2^{-1}s_1^{-1}s_2s_1^2 \in s_2^{-1}s_1^{-1}s_2u_1$, and $s_1^{-1}s_2u_1 \subset Rs_1^{-1}s_2s_1^{-1} + u_2u_1u_2$, we get

$$\begin{aligned}
s_4u_3u_2u_3s_1s_2^{-1}s_1s_4u_3u_2u_3s_4 &\subset s_4u_3u_2u_3s_2^{-1}s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\
&\subset s_4(u_3u_2u_3u_2)s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\
&\subset s_4(u_2u_3u_2u_3)s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\
&\subset u_2s_4u_3u_2u_3s_1^{-1}s_2u_1s_4u_3u_2u_3s_4 \\
&\subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4(u_3u_2u_3u_2)u_1u_2s_4u_3u_2u_3s_4 \\
&\subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_2u_3u_2u_3u_1u_2s_4u_3u_2u_3s_4 \\
&\subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_3u_2u_3u_1u_2s_4u_3u_2u_3s_4 \\
&\subset u_2s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 + A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3. We can thus restrict to $s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4$. Moreover, using that $u_3u_2u_3 \subset u_2s_3s_2^{-1}s_3 + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3^{-1}s_2s_3^{-1}u_2 + u_2u_3u_2$ leads to $s_4u_3u_2u_3s_1^{-1}s_2s_1^{-1}s_4u_3u_2u_3s_4 \subset u_2s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4u_2 + A_4u_4A_4u_4A_4$ by proposition 6.3. Now

$$\begin{aligned}
s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 &= s_1^{-1}s_1s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_4s_3(s_1s_2^{-1}s_1^{-1})s_3s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}(s_2s_3s_2)s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2s_3s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2s_1^{-1}(s_3s_4s_3^{-1})s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}(s_3s_4s_3)s_2^{-1}s_1^{-1}s_3s_2s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_4s_3s_4s_2^{-1}s_1^{-1}s_3s_2s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_4s_3s_2^{-1}s_1^{-1}(s_4s_3s_4^{-1})s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3^{-1}s_4s_3s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_4(s_3s_2^{-1}s_3^{-1})s_1^{-1}s_4s_3s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_4s_2^{-1}s_3^{-1}s_2s_1^{-1}s_4s_3s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4s_3s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4(s_3s_2s_3)s_1^{-1}s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4s_2s_3s_2s_1^{-1}s_4s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2(s_4s_3s_4)s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&= s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&\subset A_4s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4
\end{aligned}$$

and this proves the claim. \square

Lemma 6.5.

$$u_4u_3u_2u_1u_2u_3u_4u_2u_3u_1u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4$$

Proof. We consider the expression $u_4s_3^\alpha u_2u_1u_2s_3^\beta u_4u_2u_3u_1u_2u_3u_4$ and we first assume $\alpha = \beta$; by applying if necessary Φ , we can then assume $\alpha = \beta = -1$. Since $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we have

$$\begin{aligned}
u_4s_3^{-1}u_2u_1u_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 &\subset u_4s_3^{-1}u_1s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_4s_3^{-1}u_1u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 \\
&\subset u_1u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_1u_4s_3^{-1}u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4
\end{aligned}$$

and we are reduced to $u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4$ by lemmas 6.3 and 6.4. Now

$$u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 = u_4(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1u_4u_3u_2u_3u_4$$

and $(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1 \subset A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}) + A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3$ by lemma 4.4. We then have

$$\begin{aligned}
u_4(A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3)u_4u_3u_2u_3u_4 &\subset A_3u_4u_3A_3u_4u_3u_2u_3u_4 + A_3u_4s_3s_2^{-1}s_3A_3u_4u_3u_2u_3u_4 \\
&\subset A_3u_4u_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 \\
&\quad + A_3u_4s_3s_2^{-1}s_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 \\
&\subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\
&\quad + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4(u_2u_3u_2u_3)u_4 \\
&\subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\
&\quad + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_2u_4 \\
&\subset A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\
&\quad + A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_4u_2 \\
&\subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 \\
&\quad + A_4u_4A_4u_4A_4
\end{aligned}$$

by lemmas 6.3 and 6.4, and $u_4A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_4u_3u_2u_3u_4 = A_3u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3, so this solves the case $\alpha = \beta$.

We can thus assume $\alpha = -\beta$, that is we consider the expression $u_4s_3^\beta u_2u_1u_2s_3^{-\beta} u_2u_4u_1u_3u_2u_3u_4$, that we split in two cases $u_4s_3^\beta u_2u_1u_2s_3^{-\beta} s_2^\gamma u_4u_1u_3u_2u_3u_4$ for $\gamma \in \{-1, 1\}$. Up to applying Φ , we can restrict to $u_4s_3^\beta u_2u_1u_2s_3^{-\beta} s_2^\gamma s_4u_1u_3u_2u_3s_4^\alpha$ for some $\alpha \in \{-1, 1\}$, and using $u_3u_2u_3 \subset s_3^\alpha s_2^{-\alpha} s_3^\alpha u_2 + u_2u_3u_2$ we can restrict to $u_4s_3^\beta u_2u_1u_2s_3^{-\beta} s_2^\gamma s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha$ by proposition 6.3.

First assume $\gamma = -1$. Using again $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we can restrict to

$$u_4s_3^\beta s_2s_1^{-1}s_2s_3^{-\beta} s_2^{-1}s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha.$$

If $\beta = 1$, then we get

$$\begin{aligned}
u_4s_3s_2s_1^{-1}s_2s_3^{-1}s_2^{-1}s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha &\subset u_4s_3s_2s_1^{-1}(s_2s_3^{-1}s_2^{-1})s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_3s_2s_1^{-1}s_3^{-1}s_2^{-1}s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4(s_3s_2s_3^{-1})s_1^{-1}s_2^{-1}s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_2^{-1}s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset s_2^{-1}u_4s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3. For the case $\beta = -1$, we can restrict to an expression of the form

$$u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_2^{-1}s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha,$$

and we get

$$\begin{aligned}
u_4s_3^{-1}s_2s_1^{-1}(s_2s_3s_2^{-1})s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha &\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_3s_4u_1s_3^\alpha s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2(s_3s_4s_3^\alpha)u_1s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^\alpha s_3s_4u_1s_2^{-\alpha} s_3^\alpha s_4^\alpha \\
&\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^\alpha s_3u_1s_2^{-\alpha} s_4s_3^\alpha s_4^\alpha \\
&\subset u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^\alpha s_3u_1s_2^{-\alpha} s_3^\alpha s_4^\alpha s_3 \\
&\subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3 and lemma 6.4.

Now assume $\gamma = 1$. Using again $u_2u_1u_2 \subset u_1s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ we can restrict to the form $u_4s_3^\beta s_2^{-1}s_1s_2^{-1}s_3^{-\beta}s_2u_4u_1u_3u_2u_3u_4$. If $\beta = 1$ we get

$$\begin{aligned}
u_4s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2)u_4u_1u_3u_2u_3u_4 &\subset u_4s_3s_2^{-1}s_1s_3s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_4u_2(s_3^{-1}s_2s_3^{-1})s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\quad + u_4u_2u_3u_2s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_2u_4s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 + \\
&\quad u_2u_4u_3u_2s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_2u_4s_3^{-1}s_2s_1(s_3^{-1}s_2^{-1}s_3^{-1})u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\
&\subset u_2u_4s_3^{-1}(s_2s_1s_2^{-1})s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\
&\subset u_2u_4s_3^{-1}s_1^{-1}s_2s_1s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\
&\subset u_2s_1^{-1}u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\
&\subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3 and lemma 6.4.

If $\beta = -1$ we get

$$\begin{aligned}
u_4s_3^{-1}s_2^{-1}s_1(s_2^{-1}s_3s_2)u_4u_1u_3u_2u_3u_4 &\subset u_4s_3^{-1}s_2^{-1}s_1s_3s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_4(s_3^{-1}s_2^{-1}s_3)s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset u_4s_2s_3^{-1}s_2^{-1}s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset s_2u_4s_3^{-1}s_2^{-1}s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\
&\subset A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3. This concludes the proof of the lemma. \square

Proposition 6.6. $u_4A_4u_4A_4u_4 \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_4 + A_4u_4A_4u_4A_4$, and thus $A_5^{(3)} \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_4 + A_4u_4A_4u_4A_4$.

Proof. We will actually prove

$$\begin{aligned}
u_4A_4u_4A_4u_4 &\subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_4u_4A_4u_4A_4 \\
&\quad + A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3
\end{aligned}$$

and the statement will then follow by lemmas 6.4 and 6.5.

By theorem 4.1 we have $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ and $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + u_3u_2u_1u_2u_3A_3$, whence

$$\begin{aligned}
u_4A_4u_4A_4u_4 &\subset u_4A_3u_3A_3u_4A_3u_3A_3u_4 + u_4A_3u_3A_3u_4A_3u_3u_2u_3A_3u_4 \\
&\quad + u_4A_3u_3A_3u_4u_3u_2u_1u_2u_3A_3u_4 + u_4A_3u_3u_2u_3A_3u_4A_3u_3A_3u_4 \\
&\quad + u_4A_3u_3u_2u_3A_3u_4A_3u_3u_2u_3A_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3A_3u_4 \\
&\quad + u_4A_3u_3u_2u_1u_2u_3u_4A_3u_3A_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3A_3u_4 \\
&\quad + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3A_3u_4 \\
&\subset A_3u_4u_3A_3u_4u_3u_4A_3 + A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3 \\
&\subset A_3u_4u_3A_3u_4u_3u_4A_3 + A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 \\
&\quad + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3
\end{aligned}$$

We have

- (1) $A_3u_4u_3A_3u_4u_3u_4A_3 \subset A_3u_4u_3(u_2u_1u_2u_1)u_4u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by proposition 6.3.
- (2) $A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \subset A_3u_4u_3u_2u_1u_2u_1u_4u_3u_2u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by proposition 6.3.

(3) We have

$$\begin{aligned}
 A_3 u_4 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_1 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_4 u_3 (u_1 u_2 u_1 u_2) u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_4 u_3 u_2 u_1 u_2 u_1 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_4 u_4 A_4 u_4 A_4
 \end{aligned}$$

by proposition 6.3.

(4) $A_3 u_4 u_3 u_2 u_3 u_4 A_3 u_3 u_4 A_3 \subset A_3 u_4 u_3 u_2 u_3 u_4 u_2 u_1 u_2 u_1 u_3 u_4 A_3 \subset A_4 u_4 A_4 u_4 A_4$ by proposition 6.3.

(5) Using $A_3 = u_2 u_1 u_2 u_1$ we get

$$\begin{aligned}
 A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_3 u_4 A_3 &\subset A_3 u_4 (u_3 u_2 u_3 u_2) u_1 u_2 u_1 u_4 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_2 u_3 u_2 u_3 u_1 u_2 u_1 u_4 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_1 u_4 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_4 (u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4) A_3 + A_4 u_4 A_4 u_4 A_4
 \end{aligned}$$

by lemma 6.4.

(6) Using $A_3 = u_2 u_1 u_2 u_1$ we get

$$\begin{aligned}
 A_3 u_4 u_3 u_2 u_3 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 &\subset A_3 u_4 u_3 u_2 u_3 u_2 u_1 u_2 u_1 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 (u_3 u_2 u_3 u_2) u_1 u_2 u_1 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_2 u_3 u_2 u_3 u_1 u_2 u_1 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_1 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_4 u_3 u_1 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_4 u_3 (u_1 u_2 u_1 u_2) u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_4 u_3 u_2 u_1 u_2 u_1 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_3 u_1 u_2 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3
 \end{aligned}$$

(7) Using $A_3 = u_1 u_2 u_1 u_2$ we get

$$\begin{aligned}
 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_4 A_3 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_1 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_1 u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 (u_2 u_1 u_2 u_1) u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_1 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_2 u_3 u_4 A_3 \\
 &\subset A_4 u_4 A_4 u_4 A_4
 \end{aligned}$$

by proposition 6.3.

(8) Using $A_3 = u_1 u_2 u_1 u_2$ we get

$$\begin{aligned}
 A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3 u_3 u_2 u_3 u_4 A_3 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_1 u_2 u_1 u_2 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_1 u_2 u_1 (u_2 u_3 u_2 u_3) u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_1 u_2 u_1 u_3 u_2 u_3 u_2 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_1 u_2 u_1 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_1 u_3 u_4 u_2 u_1 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 (u_2 u_1 u_2 u_1) u_3 u_4 u_2 u_1 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_1 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_3 u_2 u_3 u_4 A_3 \\
 &\subset A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_2 u_1 u_3 u_2 u_3 u_4 A_3
 \end{aligned}$$

(9) the case $A_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 A_3$ is clear.

□

6.2. The A_4 -bimodule $A_5^{(3)}/A_5^{(2)}$: a smaller set of generators.

Lemma 6.7. *For all $\alpha, \beta, \gamma, \dots \in \{-1, 1\}$,*

$$\begin{aligned} s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2 s_1^{-1} s_2) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2 s_1^{-1} s_2) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2^{-1} s_1 s_2^{-1}) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2 s_1^{-1} s_2) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2 s_1^{-1} s_2) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2^{-1} s_1 s_2^{-1}) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \\ s_4^\alpha s_3^\beta A_3 s_3^\gamma s_4^\delta s_3^\varepsilon A_3 s_3^\zeta s_4^\eta &\subset u_1 s_4^\alpha s_3^\beta (s_2^{-1} s_1 s_2^{-1}) s_3^\gamma s_4^\delta s_3^\varepsilon (s_2^{-1} s_1 s_2^{-1}) s_3^\zeta s_4^\eta u_1 + A_5^{(2)} \end{aligned}$$

Proof. This is an easy consequence of the decompositions $A_3 = u_1 u_2 u_1 + u_1 s_2 s_1^{-1} s_2 = u_1 u_2 u_1 + s_2 s_1^{-1} s_2 u_1 = u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1} = u_1 u_2 u_1 + s_2^{-1} s_1 s_2^{-1} u_1$ of theorem 3.2 and of proposition 6.3. \square

Lemma 6.8. *For $i, j, k, \alpha, \beta, \gamma \in \{-1, 1\}$,*

- (1) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^\gamma s_4^k \subset A_5^{(2)}$ unless $i = j = k$
- (2) $s_4^i s_3^\alpha A_3 s_3^\beta s_4^j s_3^\gamma A_3 s_3^{-\gamma} s_4^k \subset A_5^{(2)}$ unless $i = j = k$
- (3) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^\beta s_4^k \subset A_5^{(2)}$
- (4) $s_4^i s_3^\alpha A_3 s_3^\alpha s_4^j s_3^\beta A_3 s_3^{-\beta} s_4^k \subset A_5^{(2)}$
- (5) $s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\alpha A_3 s_3^{-\alpha} s_4^k \subset A_5^{(2)}$

Proof. We use the formulas $s_3^{-1}(s_2 s_1^{-1} s_2) s_3 = s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1}$ and $s_3 (s_2 s_1^{-1} s_2) s_3^{-1} = s_2^{-1} s_1^{-1} (s_3 s_2^{-1} s_3) s_1 s_2$ which are easy to prove and which already hold in the braid group B_4 , and can be summarized as $s_3^{-\alpha} (s_2 s_1^{-1} s_2) s_3^\alpha = s_2^\alpha s_1^\alpha (s_3 s_2^{-1} s_3) s_1^{-\alpha} s_2^{-\alpha}$ for $\alpha \in \{-1, 1\}$. We also use the fact that s_2 (and thus s_2^{-1}) commutes with $s_3 s_2 s_1^{-1} s_2 s_3$ (already in the braid group B_4), and similarly s_2^{-1} (and thus s_2) commutes with $s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3^{-1}$. Together with lemma 6.7, this yields

$$\begin{aligned} s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^\beta s_4^k &\subset s_4^i s_3^\alpha (s_2 s_1^{-1} s_2) s_3^{-\alpha} s_4^j s_3^\beta (s_2 s_1^{-1} s_2) s_3^\beta s_4^k + A_5^{(2)} \\ &\subset s_4^i (s_3^\alpha s_2 s_1^{-1} s_2 s_3^{-\alpha}) s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^\beta) s_4^k + A_5^{(2)} \\ &\subset s_4^i s_2^{-\alpha} s_1^{-\alpha} (s_3 s_2^{-1} s_3) s_1^\alpha s_2^\alpha s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^\beta) s_4^k + A_5^{(2)} \\ &\subset s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^\alpha s_2^\alpha s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^\beta) s_4^k + A_5^{(2)} \\ &\subset A_3 s_4^i s_3 s_2^{-1} s_3 s_1^\alpha s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^\beta) s_2^\alpha s_4^k + A_5^{(2)} \\ &\subset A_3 s_4^i s_3 s_2^{-1} s_3 s_1^\alpha s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^\beta) s_4^k A_3 + A_5^{(2)} \\ &\subset A_5^{(2)} \end{aligned}$$

by proposition 6.3, and this proves (3), as well as the symmetric case (4). This also proves (1) in case $\beta = \gamma$. We thus deal with

$$\begin{aligned} s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^{-\beta} s_4^k &\subset s_4^i s_3^\alpha (s_2 s_1^{-1} s_2) s_3^{-\alpha} s_4^j s_3^\beta (s_2 s_1^{-1} s_2) s_3^{-\beta} s_4^k + A_5^{(2)} \\ &\subset s_4^i (s_3^\alpha s_2 s_1^{-1} s_2 s_3^{-\alpha}) s_4^j (s_3^\beta s_2 s_1^{-1} s_2 s_3^{-\beta}) s_4^k + A_5^{(2)} \\ &\subset s_4^i s_2^{-\alpha} s_1^{-\alpha} (s_3 s_2^{-1} s_3) s_1^\alpha s_2^\alpha s_4^j s_2^{-\beta} s_1^{-\beta} (s_3 s_2^{-1} s_3) s_1^\beta s_2^\beta s_4^k + A_5^{(2)} \\ &\subset s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^\alpha s_2^\alpha s_2^{-\beta} s_4^j s_1^{-\beta} (s_3 s_2^{-1} s_3) s_4^k s_1^\beta s_2^\beta + A_5^{(2)} \\ &\subset A_5^{(2)} \end{aligned}$$

if $\alpha = \beta$ by proposition 6.3, and we get (5). Otherwise, $\alpha = -\beta$, and

$$\begin{aligned} s_4^i s_3^\alpha A_3 s_3^{-\alpha} s_4^j s_3^\beta A_3 s_3^{-\beta} s_4^k &\subset s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^\alpha s_2^\alpha s_1^\alpha s_4^j (s_3 s_2^{-1} s_3) s_4^k s_1^{-\alpha} s_2^{-\alpha} + A_5^{(2)} \\ &\subset A_5^{(2)} \end{aligned}$$

unless $i = j = k$ by proposition 6.3, and we get (1). (2) is proved symmetrically. \square

Corollary 6.9.

- (1) $s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 \in A_4 u_4 A_4 u_4 A_4$
- (2) $s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 \in A_4 u_4 A_4 u_4 A_4$

Lemma 6.10. $s_4^{-1} w^+ s_4^{-1} w^+ s_4^{-1} \in A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 u_4 A_4 u_4 A_4$

Proof. We first use $s_2^{-1}s_1s_2^{-1} \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ and $s_2^{-1}s_1s_2^{-1} \in s_2s_1^{-1}s_2u_1 + u_1u_2u_1$ together with proposition 6.3 to get

$$\begin{aligned}
s_4^{-1}w^+s_4^{-1}w^+s_4^{-1} &= s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} + u_1s_4^{-1}s_3u_2u_1s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1} + A_4u_4A_4u_4A_4 \\
&\subset u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}u_1 + A_4u_4A_4u_4A_4 \\
&\subset A_4(s_3^{-1}s_4^{-1}s_3)s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2(s_3s_4^{-1}s_3^{-1})A_4 + A_4u_4A_4u_4A_4 \\
&\subset A_4s_4s_3^{-1}s_4^{-1}s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_4^{-1}s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4 \\
&\subset A_4s_4s_3^{-1}s_2s_1^{-1}s_2(s_4^{-1}s_3s_4^{-1}s_3s_4^{-1})s_2s_1^{-1}s_2s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4
\end{aligned}$$

By lemma 3.5 $s_4^{-1}s_3s_4^{-1}s_3s_4^{-1}$ is a linear combination of terms of several kinds

- (1) elements x of u_3u_4 or u_4u_3 , for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by a direct application of proposition 6.3.
- (2) elements x that can be put in the the form $s_4^\alpha s_3^\beta s_4^\gamma$ with $\alpha = -1$ or $\gamma = -1$, in which case we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ through one application of the equation $s_4s_3^{-1}s_4^{-1} \in u_3u_4u_3$ or $s_4^{-1}s_3^{-1}s_4 \in u_3u_4u_3$, and proposition 6.3.
- (3) the element $s_3^{-1}s_4s_3^{-1}$, which provides $s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4 = s_4w^-s_4w^-s_4w^-$.
- (4) the element $x = s_3s_4^{-1}s_3$, for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by corollary 6.9 (1).
- (5) the element $x = s_4^{-1}s_3s_4^{-1}s_3$, for which we get

$$\begin{aligned}
s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 &= s_4s_3^{-1}s_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&= (s_4s_3^{-1}s_4^{-1})s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&= s_3^{-1}s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&\subset A_4s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\
&\subset A_4u_4A_4u_4A_4
\end{aligned}$$

by corollary 6.9 (2).

This proves the inclusion. \square

Lemma 6.11.

- (1) $u_4A_4u_4u_3u_4 \subset A_4u_4A_4u_4A_4$
- (2) $u_4u_3u_4A_4u_4 \subset A_4u_4A_4u_4A_4$
- (3) $s_4^\beta u_3u_2u_1u_2s_3^\alpha s_4^\gamma s_3^{-\alpha} u_2u_1u_2u_3s_4^\beta \subset A_4u_4A_4u_4A_4$
- (4) $s_4^\alpha s_3^\alpha u_2u_1u_2s_3^\alpha s_4^\gamma s_3^{-\alpha} u_2u_1u_2u_3s_4^\beta \subset A_4u_4A_4u_4A_4$
- (5) $s_4^\beta u_3u_2u_1u_2s_3^\alpha s_4^\gamma s_3^{-\alpha} u_2u_1u_2s_3^{-\alpha} s_4^{-\alpha} \subset A_4u_4A_4u_4A_4$
- (6) $u_4s_3^\alpha u_2u_1u_2s_3^\alpha s_4^\gamma s_3^\alpha u_2u_1u_2s_3^\alpha u_4 \subset A_4u_4A_4u_4A_4$
- (7) $s_4w^+s_4^{-1}w^+s_4^{-1} \in A_4u_4A_4u_4A_4$
- (8) $s_4w^-s_4w^-s_4^{-1} \in A_4u_4A_4u_4A_4$

Proof. Since $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ we have $u_4A_4u_4u_3u_4 \subset A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4$. We have $u_4u_3A_3u_4u_3u_4 \subset u_4u_3u_1u_2u_1u_2u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3,

$$\begin{aligned}
u_4u_3u_2u_3A_3u_4u_3u_4 &\subset u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_4 \\
= u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_4 &= u_2u_4u_3u_2u_3u_1u_4u_2u_1u_3u_4 \subset A_4u_4A_4u_4A_4
\end{aligned}$$

by proposition 6.3, and $u_4u_3u_2u_1u_2u_3u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3. This proves (1).

(2) is deduced from (1) by applying Ψ . We turn to (3). Since $s_4^\beta u_3u_2u_1u_2(s_3^\alpha s_4^\gamma s_3^{-\alpha})u_2u_1u_2u_3s_4^\beta = s_4^\beta u_3u_2u_1u_2s_4^{-\alpha} s_3^\gamma s_4^\alpha u_2u_1u_2u_3s_4^\beta = s_4^\beta u_3s_4^{-\alpha} u_2u_1u_2s_3^\gamma u_2u_1u_2s_4^\alpha u_3s_4^\beta$ and either $s_4^\beta u_3s_4^{-\alpha} \subset u_3u_4u_3$ or $s_4^\alpha u_3s_4^\beta \subset u_3u_4u_3$. In both cases we get an element of $A_4u_4A_4u_4u_3u_4A_4 \subset A_4u_4A_4u_4A_4$ or $A_4u_4u_3u_4A_4u_4A_4 \subset A_4u_4A_4u_4A_4$ by (1) or (2), and this proves (3). (4) and (5) are similar and left to the reader.

Now

$$\begin{aligned}
u_4 s_3^\alpha u_2 u_1 u_2 s_3^\alpha s_4^\alpha s_3^\alpha u_2 u_1 u_2 s_3^\alpha u_4 &= u_4 s_3^\alpha u_2 u_1 u_2 s_4^\alpha s_3^\alpha s_4^\alpha u_2 u_1 u_2 s_3^\alpha u_4 \\
&= (u_4 s_3^\alpha s_4^\alpha) u_2 u_1 u_2 s_3^\alpha u_2 u_1 u_2 (s_4^\alpha s_3^\alpha u_4) \\
&\subset u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_2 u_1 u_2 u_3 u_4 u_3 \subset A_4 u_4 A_4 u_4 A_4
\end{aligned}$$

and this proves (6).

To prove (7), we compute, using b, b' for elements in $u_2 u_1 u_2$,

$$\begin{aligned}
s_4 w^+ s_4^{-1} w^+ s_4^{-1} &\subset s_4 s_3 b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\
&\subset s_3^{-1} (s_3 s_4 s_3) b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\
&\subset s_3^{-1} s_4 s_3 s_4 b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\
&\subset s_3^{-1} s_4 s_3 b (s_4 s_3 s_4^{-1}) s_3 b' s_3 s_4^{-1} \\
&\subset s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3^2 b' s_3 s_4^{-1}
\end{aligned}$$

Now $s_3^2 \in R + R s_3 + R s_3^{-1}$, and $s_3^{-1} s_4 s_3 b s_3^{-1} s_4 b' s_3 s_4^{-1} \in A_4 u_4 A_4 u_4 A_4$ by proposition 6.3,

$$s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3 b' s_3 s_4^{-1} \in A_4 u_4 A_4 u_4 A_4$$

by lemma 6.8 (3), and $s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3^{-1} b' s_3 s_4^{-1} \subset A_5^{(2)}$ by lemma 6.8 (1). The proof of (8) is similar :

$$\begin{aligned}
s_4 w^- s_4 w^- s_4^{-1} &\subset s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' s_3^{-1} s_4^{-1} \\
&\subset s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' (s_3^{-1} s_4^{-1} s_3^{-1}) \\
&\subset s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' s_4^{-1} s_3^{-1} s_4^{-1} \\
&\subset s_4 s_3^{-1} b s_3^{-1} (s_4 s_3^{-1} s_4^{-1}) b' s_3^{-1} s_4^{-1} \\
&\subset s_4 s_3^{-1} b s_3^{-1} s_3^{-1} s_4^{-1} s_3 b' s_3^{-1} s_4^{-1} \\
&\subset s_4 s_3^{-1} b s_3^{-2} s_4^{-1} s_3 b' s_3^{-1} s_4^{-1}
\end{aligned}$$

Now $s_3^{-2} \in R + R s_3 + R s_3^{-1}$, and we conclude similarly. □

Lemma 6.12.

- (1) $s_4 s_3^{-1} A_3 s_3 s_4 s_3 A_3 s_3^{-1} s_4 \subset u_3 s_4^- w^+ s_4 w^- s_4 + A_5^{(2)}$
- (2) $(s_4 w^+ s_4^{-1} w^+ s_4) s_3^{-1} \in s_3^{-1} (s_4 w^+ s_4^{-1} w^+ s_4) + A_5^{(2)}$
- (3) $s_4 w^+ s_4^{-1} w^+ s_4 \in A_3^\times s_4 (s_3 s_2^{-1} s_3) (s_1 s_2^{-1} s_1) s_4 (s_3 s_2^{-1} s_3) s_4 A_4^\times + A_5^{(2)}$
- (4) $s_4 w^- s_4 w^+ s_4^{-1} \in A_4^\times s_4^{-1} w^+ s_4 w^- s_4 A_4^\times + A_5^{(2)}$
- (5) $s_4 w^- s_4^{-1} w^+ s_4^{-1} \in A_4^\times s_4^{-1} w^+ s_4^{-1} w^- s_4 A_4^\times + A_5^{(2)}$
- (6) $s_4 s_3 A_3 s_3^{-1} s_4 s_3^{-1} A_3 s_3 s_4 \subset u_3 s_4 w^+ s_4^{-1} w^+ s_4 u_3 + A_5^{(2)}$

Proof. We first prove (1). By lemma 6.7 we need to prove $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \subset u_3 s_4^- w^+ s_4 w^- s_4 + A_5^{(2)}$, and we get, using proposition 6.3

$$\begin{aligned}
s_4 s_3^{-1} s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_2 s_1^{-1} s_2 s_3^{-1} s_4 &= s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\
&= (s_4 s_3^{-1} s_4) s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\
&\subset u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\
&\quad + u_3 u_4 u_3 s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\
&\subset u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\
&\subset u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_4^{-1} s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\
&\subset u_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\
&\subset u_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\
&\subset u_3 s_4^{-1} w^+ s_4 w^- s_4 + A_5^{(2)}.
\end{aligned}$$

We now prove (2). We have, using $s_3 s_4^{-1} s_3 s_4^{-1} \in R s_4 s_3^{-1} s_4 s_3^{-1} + u_3 u_4 u_3 + u_4 u_3 u_4$ and $s_3 s_4^{-1} s_3 s_4^{-1} - s_4^{-1} s_3 s_4^{-1} s_3 \in u_3 u_4 + u_4 u_3$ by lemma 3.6, we get

$$\begin{aligned}
s_4 w^+ s_4^{-1} w^+ s_4 \cdot s_3^{-1} &= s_4 w^+ s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} (s_3 s_4 s_3^{-1}) \\
&= s_4 w^+ s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_4^{-1} s_3 s_4 \\
&= s_4 w^+ s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_2^{-1} s_3 s_4 \\
&= s_4 s_3 s_2^{-1} s_1 s_2^{-1} (s_3 s_4^{-1} s_3 s_4^{-1}) s_2^{-1} s_1 s_2^{-1} s_3 s_4 \\
&\in s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_4^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
&\subset (s_4 s_3 s_4^{-1}) s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
&\subset s_3^{-1} s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
&\subset s_3^{-1} \cdot s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}
\end{aligned}$$

We now prove (3). We have

$$\begin{aligned}
s_4 w^+ s_4^{-1} w^+ s_4 &= s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4 \\
&\in s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4 + A_5^{(2)} \\
&\subset s_3^{-1} (s_3 s_4 s_3) s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_3^{-1} + A_5^{(2)} \\
&\subset s_3^{-1} s_4 s_3 s_4 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_4 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3 (s_4^{-1} s_3 s_4) s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 (s_3^2) s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4^2 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
&\quad + R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 (s_4 s_3 s_4) s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
&\quad + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
&\quad + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
&\quad + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1} (s_4 s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_3^{-1} s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_4 (s_3 s_2 s_1^{-1} s_2 s_3^{-1}) s_4 (s_3^{-1} s_2 s_1^{-1} s_2 s_3) s_4 s_3^{-1} + A_5^{(2)} \\
&\subset R^\times s_4 s_2^{-1} s_1^{-1} (s_3 s_2^{-1} s_3) s_1 s_2 s_4 s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1} s_4 s_3^{-1} + A_5^{(2)} \\
&\subset A_3^\times s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_2 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 s_1^{-1} s_2^{-1} s_3^{-1} + A_5^{(2)}
\end{aligned}$$

and then $s_1 s_2 s_2 s_1 = s_1 s_2^2 s_1 \in R^\times s_1 s_2^{-1} s_1 + R s_1 s_2 s_1 + R s_1^2$. Since $s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 = s_4 (s_3 s_2^{-1} s_3) s_2 s_1 s_2 s_4 (s_3 s_2^{-1} s_3) s_4 \subset s_4 (u_3 u_2 u_3 u_2) s_1 s_2 s_4 u_3 u_2 u_3 s_4 = s_4 u_2 u_3 u_2 u_3 s_1 s_2 s_4 u_3 u_2 u_3 s_4 = u_2 s_4 u_3 u_2 u_3 s_1 s_2 s_4 u_3 u_2 u_3 s_4 \subset A_5^{(2)}$ by proposition 6.3 and similarly $s_4 (s_3 s_2^{-1} s_3) s_1 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 \in s_4 (s_3 s_2^{-1} s_3) u_1 s_4 (s_3 s_2^{-1} s_3) s_4 \subset A_5^{(2)}$, this proves

$$s_4 w^+ s_4^{-1} w^+ s_4 \in A_3^\times s_4 (s_3 s_2^{-1} s_3) (s_1 s_2^{-1} s_1) s_4 (s_3 s_2^{-1} s_3) s_4 A_4^\times + A_5^{(2)}.$$

We now prove (4).

$$\begin{aligned}
s_4 w^- s_4 w^+ s_4^{-1} &= s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1}) \\
&= s_3^{-1} (s_3 s_4 s_3^{-1}) (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1}) s_3 \\
&= s_3^{-1} (s_4^{-1} s_3 s_4) (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} s_4^{-1} s_3^{-1} s_4) s_3 \\
&= s_3^{-1} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 s_3^{-1} s_4 s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_4 s_3)
\end{aligned}$$

We now prove (4).

$$\begin{aligned}
s_4 w^- s_4 w^+ s_4^{-1} &= s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} \\
&= s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} \\
&= s_3^{-1} (s_3 s_4 s_3^{-1}) (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) (s_3 s_4^{-1} s_3^{-1}) s_3 \\
&= s_3^{-1} s_4^{-1} s_3 s_4 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_4^{-1} s_3^{-1} s_4 s_3 \\
&= s_3^{-1} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 s_3^{-1} s_4 s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 s_3
\end{aligned}$$

We have $s_4 s_3^{-1} (s_4 s_3 s_4^{-1}) = s_4 s_3^{-1} s_3^{-1} s_4 s_3 = s_4 s_3^{-2} s_4 s_3 \in R^\times s_4 s_3 s_4 s_3 + u_4 s_3 + R s_4 s_3^{-1} s_4 s_3 = R^\times (s_4 s_3 s_4) s_3 + u_4 s_3 + R s_4 s_3^{-1} s_4 s_3 = R^\times s_3 s_4 s_3^2 + u_4 s_3 + R s_4 s_3^{-1} s_4 s_3 \subset R^\times s_3 s_4 s_3^{-1} + R s_3 s_4 s_3 + R s_3 s_4 + u_4 s_3 + R s_4 s_3^{-1} s_4 s_3$. We have

$$s_4^{-1} s_3 (s_2 s_1^{-1} s_2) (u_4 s_3) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \subset A_5^{(2)}$$

by lemma 6.11 (2) ;

$$s_4^{-1} s_3 (s_2 s_1^{-1} s_2) (s_3 s_4) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \subset A_5^{(2)}$$

by lemma 6.11 (1) ;

$$s_4^{-1} s_3 (s_2 s_1^{-1} s_2) (s_3 s_4 s_3) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \subset A_5^{(2)}$$

by lemma 6.8 (4) ;

$$\begin{aligned} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) (s_4 s_3^{-1} s_4 s_3) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 &= s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 (s_3^{-1} s_4 s_3) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \\ &= s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 (s_4 s_3 s_4^{-1}) (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \\ &= s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4^2 s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 \\ &= s_4^{-1} s_3 s_4^2 (s_2 s_1^{-1} s_2) s_3 (s_2^{-1} s_1 s_2^{-1}) (s_4^{-1} s_3^{-1} s_4) \\ &= s_4^{-1} s_3 s_4^2 (s_2 s_1^{-1} s_2) s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} s_3^{-1} \\ &\in A_5^{(2)} \end{aligned}$$

by lemma 6.11. It follows that $s_4 w^- s_4 w^+ s_4^{-1} \in u_3^\times s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4 s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 u_3^\times + A_5^{(2)}$ hence $s_4 w^- s_4 w^+ s_4^{-1} \in u_3^\times s_4^{-1} w^+ s_4 w^- s_4 u_3^\times + A_5^{(2)}$

The proof of (5) is similar : one first gets

$$s_4 w^- s_4^{-1} w^+ s_4^- = s_3^{-1} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 s_3^{-1} s_4^{-1} s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 s_3$$

and then writes down $(s_4 s_3^{-1} s_4^{-1}) s_3 s_4^{-1} = s_3^{-1} s_4^{-1} s_3^2 s_4^{-1} \in R^\times s_3^{-1} (s_4^{-1} s_3^{-1} s_4^{-1}) + R (s_3^{-1} s_4^{-1} s_3) s_4^{-1} + R s_3^{-1} s_4^{-2} = R^\times s_3^{-2} s_4^{-1} s_3^{-1} + R s_4 s_3^{-1} s_4^{-2} + R s_3^{-1} s_4^{-2} \subset R^\times s_3 s_4^{-1} s_3^{-1} + R s_3^{-1} s_4^{-1} s_3^{-1} + R s_4^{-1} s_3^{-1} + R s_4 s_3^{-1} u_4 + R s_3^{-1} s_4^{-2}$; one then shows using the same arguments as before that all terms but $R^\times s_3 s_4^{-1} s_3^{-1}$ provide an element of $A_5^{(2)}$, thus

$$\begin{aligned} s_4 w^- s_4^{-1} w^+ s_4^- &\in u_3^\times s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4^{-1} s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 u_3^\times + A_5^{(2)} \\ &\in u_3^\times s_4^{-1} w^+ s_4^{-1} w^- s_4 u_3^\times + A_5^{(2)} \end{aligned}$$

We prove (6).

$$\begin{aligned} s_4 s_3 A_3 s_3^{-1} s_4 s_3^{-1} A_3 s_3 s_4 &\subset s_4 s_3 A_3 s_3^{-1} s_4 s_3^{-1} A_3 (s_3 s_4 s_3) s_3^{-1} \\ &\subset s_4 s_3 A_3 s_3^{-1} s_4 s_3^{-1} A_3 s_4 s_3 s_4 s_3^{-1} \\ &\subset s_4 s_3 A_3 (s_3^{-1} s_4 s_3^{-1} s_4) A_3 s_3 s_4 s_3^{-1} \\ &\subset s_4 s_3 A_3 s_4^{-1} s_3 s_4^{-1} s_3 A_3 s_3 s_4 s_3^{-1} + A_5^{(2)} \quad (\text{lemmas 3.6 and 6.11 (1)}) \\ &\subset (s_4 s_3 s_4^{-1}) A_3 s_3 s_4^{-1} s_3 A_3 s_3 s_4 s_3^{-1} + A_5^{(2)} \\ &\subset s_3^{-1} s_4 s_3 A_3 s_3 s_4^{-1} s_3 A_3 s_3 s_4 s_3^{-1} + A_5^{(2)} \\ &\subset u_3 s_4 w^+ s_4^{-1} w^+ s_4 u_3 + A_5^{(2)} \quad (\text{lemma 6.7}) \end{aligned}$$

□

Proposition 6.13.

$$\begin{aligned} u_4 u_3 u_2 u_1 u_2 u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 &\subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 \\ &\quad + A_4 s_4 w^- s_4 w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4^{-1} w^+ s_4^{-1} A_4 + A_5^{(2)} \end{aligned}$$

Proof. We first note that, by lemma 6.10 and lemma 6.12 (4) and (5), the right-hand side (RHS) of the statement is invariant under Φ and Ψ . We now consider an expression of the form $s_4^\alpha s_3^\beta u_2 u_1 u_2 s_3^\gamma s_4^\delta u_2 u_1 u_2 s_3^\alpha s_4^\delta$ with $\alpha, \beta, \gamma \in \{-1, 1\}$. By lemma 6.8 we can assume $\alpha = \beta$ and $\gamma = \delta$, except for the expression $s_4^\alpha s_3^\alpha u_2 u_1 u_2 s_3^{-\alpha} s_4^\alpha s_3^{-\alpha} u_2 u_1 u_2 s_3^\alpha s_4^\alpha$. Up to applying Φ , we can moreover assume $\varepsilon = 1$, and we get the conclusion by lemma 6.12 (6) for $\alpha = 1$, by lemma 6.12 (1) and (4) for $\alpha = -1$.

We can now assume $\alpha = \beta$, $\gamma = \delta$, and still $\varepsilon = 1$. By lemma 6.7 this reduces our examination to expressions $x = s_3 w^\alpha s_4^\varepsilon w^\beta s_4^\eta$ for new parameters $\alpha, \varepsilon, \beta, \eta \in \{-1, 1\}$. If $\alpha = \beta = \varepsilon$, we have

$x \in A_5^{(2)}$ by lemma 6.11 (6) ; if $\alpha = \beta = -\varepsilon$, we get $x \in A_5^{(2)}$ if in addition $\eta = -1$, by lemma 6.11 (7) and (8), and $x = s_4 w^\alpha s_4^{-\alpha} w^\alpha s_4 \in RHS$ otherwise. As a consequence, we can reduce to the case $\alpha = -\beta$, that is $x = s_4 w^\alpha s_4^\varepsilon w^{-\alpha} s_4^\eta$. If $\alpha = 1$, $x \in A_5^{(2)}$ by lemma 6.11 (4). If $\alpha = -1$, all the possibilities for x clearly lie in the RHS, except for $s_4 w^- s_4^{-1} w^+ s_4$, which belongs to $A_5^{(2)}$ by lemma 6.11 (3). This concludes the proof. \square

6.3. Image of the center of the braid group in $A_5^{(3)}/A_5^{(2)}$. Recall that the center of the braid group B_n is infinite cyclic, generated for $n \geq 3$ by $c_n = (s_1 \dots s_{n-1})^n$, and that this generator can be written as $c_n = c_{n-1} y_n = y_n c_{n-1} = y_n y_{n-1} \dots y_3 y_2$ where the $y_n \in B_n \setminus B_{n-1}$ under the usual inclusions $B_2 \subset B_3 \subset \dots \subset B_{n-1}$ form another family of commuting elements defined by $y_2 = s_1^2$ and $y_{n+1} = s_n y_n s_n = s_n s_{n-1} \dots s_2 s_1^2 s_2 \dots s_{n-1} s_n$.

We let $c = c_5 = (s_1 s_2 s_3 s_4)^5 = (s_4 s_3 s_2 s_1)^5$. The center of G_{32} is cyclic of order 6 and is generated by the image of c . We let $w_0 = y_4 = s_3 s_2 s_1^2 s_2 s_3 = c_4 c_3^{-1}$, which by definition commutes with B_3 , and $\delta = y_5 = s_4 s_3 s_2 s_1^2 s_2 s_3 s_4 = c_5 c_4^{-1}$ which commutes with B_4 .

We first need a preparatory lemma.

Lemma 6.14.

- (1) In A_4 , $s_4^\alpha w_0^\beta s_4^\gamma w_0 s_4^\delta \in A_3^\times s_4^\alpha w_0^{-1} s_4^\beta w_0 s_4^\gamma + A_3 s_4^\alpha w_0 s_4^\beta w_0 s_4^\gamma + A_5^{(2)}$
- (2) For all $\alpha, \beta, \gamma, \delta, \varepsilon \in \{-1, 1\} = \{-, +\}$,

$$\begin{aligned} s_4^\alpha w^\beta s_4^\gamma w^\delta s_4^\varepsilon &\in s_4^\alpha w^\beta s_4^\gamma w_0^\delta s_4^\varepsilon A_3^\times + s_4^\alpha w^\beta s_4^\gamma u_1 u_3 u_2 u_3 s_4^\varepsilon A_3 \\ &\subset s_4^\alpha w^\beta s_4^\gamma w_0^\delta s_4^\varepsilon A_3^\times + A_5^{(2)} \\ s_4^\alpha w^\beta s_4^\gamma w^\delta s_4^\varepsilon &\in A_3^\times s_4^\alpha w_0^\beta s_4^\gamma w^\delta s_4^\varepsilon + A_3 s_4^\alpha u_3 u_2 u_3 u_1 s_4^\gamma w^\delta s_4^\varepsilon \\ &\subset A_3^\times s_4^\alpha w_0^\beta s_4^\gamma w^\delta s_4^\varepsilon + A_5^{(2)} \\ s_4^\alpha w^\beta s_4^\gamma w^\delta s_4^\varepsilon &\in A_3^\times s_4^\alpha w_0^\beta s_4^\gamma w_0^\delta s_4^\varepsilon + A_3 s_4^\alpha u_3 u_2 u_3 u_1 s_4^\gamma w^\delta s_4^\varepsilon + A_3 s_4^\alpha w_0^\gamma s_4^\gamma u_3 u_2 u_3 u_1 s_4^\varepsilon \\ &\subset A_3^\times s_4^\alpha w_0^\beta s_4^\gamma w_0^\delta s_4^\varepsilon + A_5^{(2)} \end{aligned}$$

Proof. (1) is a straightforward consequence of lemma 4.9 and of the fact that $s_4^\alpha U_0 s_4^\beta w_0 s_4^\gamma \subset s_4^\alpha A_3 u_3 A_3 s_4^\beta w_0 s_4^\gamma + s_4^\alpha A_3 u_3 u_2 u_3 s_4^\beta w_0 s_4^\gamma = A_3 s_4^\alpha u_3 s_4^\beta w_0 s_4^\gamma A_3 + A_3 s_4^\alpha u_3 u_2 u_3 s_4^\beta w_0 s_4^\gamma A_3 \subset A_5^{(2)}$ by proposition 6.3. (2) follows from an easy variation in the proof of lemma 6.7 and from lemma 4.6. \square

We are then in position to prove the following.

Lemma 6.15.

- (1) $s_4 w^- s_4 w^+ s_4^{-1} \in A_4^\times s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}$
- (2) $s_4 w^- s_4^{-1} w^+ s_4^{-1} \in s_4^{-1} w^- s_4 w^- s_4^{-1} A_4^\times + A_5^{(2)}$

Proof. We have $s_4 w^- s_4 w^+ s_4^{-1} \in A_3^\times s_4 w_0^{-1} s_4 w_0 s_4^{-1} + A_5^{(2)}$ by lemma 6.14 (2). Since $s_4 w_0^{-1} s_4 w_0 s_4 \in A_3^\times s_4 w^- s_4 w^+ s_4 \subset A_5^{(2)}$ by lemma 6.11 (5) and since $s_4^{-1} \in R^\times s_4^2 + R s_4 + R$, we have $s_4 w_0^{-1} s_4 w_0 s_4^{-1} \equiv s_4 w_0^{-1} s_4 w_0 s_4^2 \pmod{A_5^{(2)}}$. Then $s_4 w_0^{-1} s_4 w_0 s_4^2 \in A_3^\times s_4 w_0^2 s_4 w_0 s_4^2 + A_3 s_4 w_0 s_4 w_0 s_4^2 + A_5^{(2)}$ by lemma 6.14 (1), and $s_4 w_0 s_4 w_0 s_4^2 \in R s_4 w_0 s_4 w_0 s_4 + R s_4 w_0 s_4 w_0 s_4^{-1} + A_5^{(2)}$. Then

$$\begin{aligned} s_4 w_0 s_4 w_0 s_4^{-1} &\in A_3^\times s_4 w^+ s_4 w^+ s_4^{-1} \subset A_5^{(2)} \\ \text{and } s_4 w_0 s_4 w_0 s_4 &\in A_3^\times s_4 w^+ s_4 w^+ s_4 \subset A_5^{(2)} \end{aligned}$$

by lemmas 6.14 (2) and 6.11 (6). It follows that $s_4 w_0^{-1} s_4 w_0 s_4^2 \in A_3^\times s_4 w_0^2 s_4 w_0 s_4^2 + A_5^{(2)}$.

Now $s_4 w_0^2 s_4 w_0 s_4^2 = s_4 w_0 (w_0 (s_4 w_0 s_4)) s_4$ and $w_0 (s_4 w_0 s_4) = c_2^{-1} c_4 \in A_3^\times c_4$ commutes with w_0 and s_4 . Thus $s_4 w_0^2 s_4 w_0 s_4^2 = (w_0 (s_4 w_0 s_4)) s_4 w_0 s_4 \in A_4^\times s_4 w_0 s_4^2 w_0 s_4$. Now $s_4 w_0 s_4^2 w_0 s_4 \in R^\times s_4 w_0 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4 w_0 s_4 + A_5^{(2)}$; moreover we already noticed $s_4 w_0 s_4 w_0 s_4 \in A_5^{(2)}$, hence $s_4 w_0 s_4^2 w_0 s_4 \in R^\times s_4 w_0 s_4^{-1} w_0 s_4 + A_5^{(2)} \subset A_3^\times s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}$ by lemma 6.14 (2), and this proves (1).

Now we have $s_4 w^- s_4^{-1} w^+ s_4^{-1} = \Psi(s_4 w^- s_4 w^+ s_4^{-1}) \in \Psi(A_4^\times s_4 w^+ s_4^{-1} w^+ s_4) + \Psi(A_5^{(2)}) = s_4^{-1} w^- s_4 w^- s_4^{-1} A_4^\times + A_5^{(2)}$, and this proves (2). \square

By a direct computation, we will prove the following lemma, which will turn out to be crucial in the proof of the main theorem. We postpone this (lengthy) calculation to section 7.

Lemma 6.16. *In A_5 , δ^3 belongs to*

$$A_4^\times s_4 w^- s_4 w^- s_4 A_3^\times + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$$

6.4. Right actions are left actions.

Lemma 6.17.

(1) *For all $\alpha, \beta, \gamma, \dots \in \{-1, 1\}$, $x, y \in A_3$,*

$$s_1(s_4^\alpha s_3^\beta x s_3^\gamma s_4^\delta s_3^\epsilon y s_3^\zeta s_4^\eta) \equiv (s_4^\alpha s_3^\beta x s_3^\gamma s_4^\delta s_3^\epsilon y s_3^\zeta s_4^\eta) s_1 \pmod{A_5^{(2)}}$$

(2) *For all $x \in A_4$, $(s_4 w^+ s_4^{-1} w^+ s_4) x \in A_4 (s_4 w^+ s_4^{-1} w^+ s_4) \pmod{A_5^{(2)}}$*

(3) *For all $x \in A_4$, $(s_4^{-1} w^- s_4 w^- s_4^{-1}) x \in A_4 (s_4^{-1} w^- s_4 w^- s_4^{-1}) \pmod{A_5^{(2)}}$*

(4) *$(s_4 w^- s_4 w^+ s_4^{-1}) s_3^{-1} \in s_3^{-1} (s_4 w^- s_4 w^+ s_4^{-1}) + A_5^{(2)}$*

(5) *$(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1} \in u_3 s_4^{-1} w^+ s_4^{-1} w^- s_4 + A_5^{(2)}$.*

Proof. We first prove (1). By lemma 6.7 and because s_1 commutes with u_1 we can assume $x = y = s_2^{-1} s_1 s_2^{-1}$. Since $(s_2^{-1} s_1 s_2^{-1}) s_1 \in s_1 (s_2^{-1} s_1 s_2^{-1}) + u_1 u_2 u_1$ by lemma 2.3 (and even $(s_2^{-1} s_1 s_2^{-1}) s_1 \in s_1 (s_2^{-1} s_1 s_2^{-1}) + u_1 u_2 + u_2 u_1$, see lemma 3.6), by proposition 6.3 we get the conclusion.

We then prove (2). Because of (1), and because we have the result for $x = s_3^{-1}$ by lemma 6.12 (2), we need only consider $x = s_2$. For $x = s_2$, we first use that

$$s_4 w^+ s_4^{-1} w^+ s_4 \in u_1^\times s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4 u_1^\times + A_5^{(2)}$$

by lemma 6.7 ; then, because of (1) we get that

$$u_1^\times s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4 u_1^\times \subset u_1^\times s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4 + A_5^{(2)}.$$

Then

$$\begin{aligned} s_4 w^+ s_4^{-1} w^+ s_4 \cdot s_2 &\in u_1^\times s_4 (s_3 s_2 s_1^{-1} s_2 s_3) s_4^{-1} (s_3 s_2 s_1^{-1} s_2 s_3) s_4 s_2 + A_5^{(2)} \\ &\subset u_1^\times s_2 s_4 (s_3 s_2 s_1^{-1} s_2 s_3) s_4^{-1} (s_3 s_2 s_1^{-1} s_2 s_3) s_4 + A_5^{(2)} \end{aligned}$$

because s_2 commutes with both s_4 and $s_3 s_2 s_1^{-1} s_2 s_3$ and this proves (2). One gets (3) by applying Φ to (2).

We prove (4). One easily gets $(s_4 w^- s_4 w^+ s_4^{-1}) s_3^{-1} = s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4$. Now $s_4^{-2} \in R^\times s_4 + R s_4^{-1} + R$, and it is easily checked that the terms originating from $R s_4^{-1}$ and R belong to $A_5^{(2)}$. We thus get $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in s_4 s_3^{-1} s_2 s_1^{-1} s_2 (s_4 s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \subset s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)}$, and

$$s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in u_3 s_4^{-1} w^+ s_4 w^- s_4 + A_5^{(2)}$$

by lemma 6.12 (1). We prove (5). One easily gets

$$(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1} = s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4,$$

and $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} x s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in A_5^{(2)}$ for $x \in 1, s_4^{-1}$, hence $(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1}$ belongs to

$$\begin{aligned} &s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} = s_4 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} \\ &\subset u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} = u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_4^{-1} s_3^{-1} s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} \\ &= u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} \subset u_3 s_4^{-1} w^+ s_4^{-1} w^- s_4 + A_5^{(2)} \end{aligned}$$

□

Remark 6.18. Another proof of item (2). It is easily checked that $s_4 w^+ s_4^{-1} w^+ s_4 \equiv (s_4 s_3 s_2 s_1^2 s_2 s_3 s_4)^2 \pmod{A_5^{(2)}}$, and the element $s_4 s_3 s_2 s_1^2 s_2 s_3 s_4$ of the braid group B_5 is well-known to centralize B_4 .

Proposition 6.19.

$$\begin{aligned} A_5^{(3)} &= A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} \\ A_5^{(3)} &= s_4 w^- s_4 w^- s_4 A_4 + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} \end{aligned}$$

Proof. Clearly the RHS are included in $A_5^{(3)}$. By propositions 6.6 and 6.13 we have $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_4 s_4 w^- s_4 w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4^{-1} w^+ s_4^{-1} A_4 + A_5^{(2)}$. Lemma 6.15 then implies $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$. By lemma 6.17 this implies $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)}$ and $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$. Now, by lemma 6.16, $s_4 w^- s_4 w^- s_4$ belongs to $A_4^\times \delta^3 A_3^\times + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$, hence $A_4 s_4 w^- s_4 w^- s_4 A_4 \subset A_4 \delta^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} = A_4 c^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} = A_4 c^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$ and, since c^3 is central and by lemma 6.17, this latter expression can be written as $A_4 c^3 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 \delta^3 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 s_4 w^- s_4 w^- s_4 A_3^\times + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)}$. The other expression is deduced from this one by application of $\Phi \circ \Psi$. \square

Proposition 6.20. $A_5 = A_5^{(3)}$.

Proof. One only needs to prove $A_5^{(4)} \subset A_5^{(3)}$, that is $u_4 a u_4 b u_4 c u_4 \subset A_5^{(3)}$ for all $a, b, c \in A_4$. We have $u_4 a u_4 b u_4 c \in A_5^{(3)} = A_5^{(2)} A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1}$ hence

$$u_4 a u_4 b u_4 c u_4 \subset A_5^{(2)} u_4 A_4 s_4 w^- s_4 w^- s_4 u_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 u_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} u_4 \subset A_5^{(3)}.$$

This proves the claim. \square

This proves theorem 6.1, and actually the following refinement :

Theorem 6.21.

$$\begin{aligned} A_5 &= A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 \\ &\quad + A_4 s_4^{-1} w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} A_4 + A_4 s_4 w^+ s_4 A_4 + s_4 w^- s_4 w^- s_4 A_4 \\ &\quad + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 \end{aligned}$$

6.5. A_5 as a A_4 -module. We need the following lemma on A_3 :

Lemma 6.22.

- (1) $u_2 u_1 u_2 \subset u_1 s_2 s_1^2 s_2 + u_1 u_2 u_1$
- (2) $u_2 u_1 u_2 \subset u_1 s_2^{-1} s_1^{-2} s_2^{-1} + u_1 u_2 u_1$

Proof. (2) is a consequence of (1) by using Φ , so it is enough to prove (1). We have $u_2 u_1 u_2 \subset u_1 u_2 u_1 + \sum_{\alpha \in \{-1, 1\}} R s_2^\alpha s_1^{-\alpha} s_2^\alpha$ because u_i is R -spanned by $1, s_i, s_i^{-1}$ and because of lemma 2.2. Moreover $s_2^{-1} s_1 s_2^{-1} \in u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$ by lemmas 2.4 and 2.3, hence $u_2 u_1 u_2 \subset u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$. Since $s_1^{-1} \in R s_1^2 + R s_1 + R$ we get $s_2 s_1^{-1} s_2 \in R s_2 s_1^2 s_2 + R s_2 s_1 s_2 + R s_2^2 \subset R s_2 s_1^2 s_2 + u_1 u_2 u_1$ hence $u_2 u_1 u_2 \subset u_1 s_2 s_1^2 s_2 + u_1 u_2 u_1$. \square

We introduce or re-introduce the following submodules of A_5 :

$$\begin{aligned} A_5^{(1)} &= A_4 u_4 A_4 = A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 \\ A_5^{(1\frac{1}{4})} &= A_5^{(1)} + A_4 s_4 s_3^{-1} s_4 A_4 (= A_5^{(1)} + A_4 s h^2(A_3) A_4) \\ A_5^{(1\frac{1}{2})} &= A_5^{(1\frac{1}{4})} + A_4 u_4 u_3 u_2 u_3 u_4 A_4 \\ &= A_5^{(1\frac{1}{4})} + A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 \\ A_5^{(2)} &= A_4 u_4 A_4 u_4 A_4 = A_5^{(1\frac{1}{2})} + \sum_{\alpha, \beta \in \{1, -1\}} A_4 s_4^\alpha w^\beta s_4^\alpha A_4 \end{aligned}$$

and

$$A_5 = A_5^{(3)} = A_4 u_4 A_4 u_4 A_4 u_4 A_4 = A_5^{(2)} + A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1}$$

We let \mathcal{B} denote the family of elements defined in corollary 5.12, which span A_4 as a left B -module, \mathcal{A} the family spanning A_4 as a A_3 -module defined in proposition 4.8, and \mathcal{A}' its image under the automorphism $\text{Ad } \Delta$ of A_4 (that is $s_1 \leftrightarrow s_3$, $s_2 \leftrightarrow s_2$). We prove the following.

Lemma 6.23.

- (1) $A_5^{(1)} = A_4 + \sum_{x \in \mathcal{A}} A_4 s_4 x + \sum_{x \in \mathcal{A}} A_4 s_4^{-1} x$
- (2) $A_5^{(1\frac{1}{4})} = A_5^{(1)} + \sum_{x \in \mathcal{B}} A_4 s_4 s_3^{-1} s_4 x$

Proof. (1) is a consequence of $A_3 s_4^{\pm 1} = s_4^{\pm 1} A_3$, because

$$A_5^{(1)} = A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 = A_4 + A_4 s_4 \sum_{x \in \mathcal{A}} A_3 x + A_4 s_4^{-1} \sum_{x \in \mathcal{A}} A_3 x = A_4 + \sum_{x \in \mathcal{A}} A_4 s_4 x + \sum_{x \in \mathcal{A}} A_4 s_4^{-1} x$$

We prove (2). We have $(s_4 s_3^{-1} s_4) s_1 = s_1 (s_4 s_3^{-1} s_4)$ and $(s_4 s_3^{-1} s_4) s_3^{-1} \in s_3^{-1} (s_4 s_3^{-1} s_4) + u_3 u_4 + u_4 u_3$ by lemma 3.6, hence $(s_4 s_3^{-1} s_4) B \subset B (s_4 s_3^{-1} s_4) + A_5^{(1)}$, where we recall $B = \langle s_1, s_3^{-1} \rangle = \langle s_1, s_3 \rangle$. Thus

$$A_5^{(1\frac{1}{4})} = A_5^{(1)} + A_4 s_4 s_3^{-1} s_4 \sum_{x \in \mathcal{B}} Bx = A_5^{(1)} + \sum_{x \in \mathcal{B}} A_4 s_4 s_3^{-1} s_4 Bx = A_5^{(1)} + \sum_{x \in \mathcal{B}} A_4 s_4 s_3^{-1} s_4 x$$

□

Lemma 6.24.

- (1) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + A_4 s_4 s_3 s_2^2 s_3 s_4 A_4 + A_4 s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} A_4$
- (2) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + \sum_{x \in \mathcal{A}'} A_4 s_4 s_3 s_2^2 s_3 s_4 x + \sum_{x \in \mathcal{A}'} A_4 s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} x$

Proof. We have $s_3^{-1} s_2 s_3^{-1} \subset u_2 s_3 s_2^2 s_3 + u_2 u_3 u_2$ by lemma 6.22 hence $s_4 s_3^{-1} s_2 s_3^{-1} s_4 \subset s_4 u_2 s_3 s_2^2 s_3 s_4 + s_4 u_2 u_3 u_2 s_4 \subset u_2 s_4 s_3 s_2^2 s_3 s_4 + u_2 s_4 u_3 s_4 u_2 \subset u_2 s_4 s_3 s_2^2 s_3 s_4 + A_5^{(1\frac{1}{4})}$. Applying Φ this implies $s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} \subset u_2 s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} + A_5^{(1\frac{1}{4})}$ which proves (1). Let $A'_3 = \langle s_2, s_3 \rangle$. We have $A_4 = \sum_{x \in \mathcal{A}'} x$. Since $s_4 s_3 s_2^2 s_3 s_4$ commutes with s_2 and s_3 hence to A'_3 , we get

$$A_4 s_4 s_3 s_2^2 s_3 s_4 A_4 \subset \sum_{x \in \mathcal{A}'} A_4 s_4 s_3 s_2^2 s_3 s_4 A'_3 x \subset \sum_{x \in \mathcal{A}'} A_4 s_4 s_3 s_2^2 s_3 s_4 x$$

and similarly $s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} = (s_4 s_3 s_2^2 s_3 s_4)^{-1}$ commutes with s_2 and s_3 hence

$$A_4 s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} A_4 \subset \sum_{x \in \mathcal{A}'} A_4 s_4^{-1} s_3^{-1} s_2^{-2} s_3^{-1} s_4^{-1} x$$

which proves (2). □

Lemma 6.25.

$$A_5^{(2)} = A_5^{(1\frac{1}{2})} + \sum_{\alpha \in \{-1, 1\}} A_4 s_4^\alpha w^\alpha s_4^\alpha + \sum_{\alpha \in \{-1, 1\}} \sum_{x \in \mathcal{A}} A_4 s_4^\alpha w^{-\alpha} s_4^\alpha x$$

Proof. By lemma 4.6 (1), we have $w_0 \in A_3^\times w^+ + U_0$, $w_0^{-1} \in A_3^\times w^- + U_0$, hence $w^+ \in A_3^\times w_0 + U_0$, $w^- \in A_3^\times w_0^{-1} + U_0$, with $U_0 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3 \subset A_4$. As a consequence, for $\alpha, \beta \in \{-1, 1\}$, we have $A_4 s_4^\alpha w^\beta s_4^\alpha A_4 \subset A_4 s_4^\alpha A_3^\times w_0^\beta s_4^\alpha A_4 + A_4 s_4^\alpha U_0 s_4^\alpha A_4$. Moreover, $s_4^\alpha U_0 s_4^\alpha = s_4^\alpha A_3 u_3 A_3 s_4^\alpha + s_4^\alpha A_3 u_3 u_2 u_3 A_3 s_4^\alpha = A_3 s_4^\alpha u_3 s_4^\alpha A_3 + A_3 s_4^\alpha u_3 u_2 u_3 s_4^\alpha A_3 \subset A_5^{(1\frac{1}{4})} + A_5^{(1\frac{1}{2})} = A_5^{(1\frac{1}{2})}$, hence $A_4 s_4^\alpha w^\beta s_4^\alpha A_4 \subset A_4 s_4^\alpha w_0^\beta s_4^\alpha A_4 + A_5^{(1\frac{1}{2})}$. Since w_0 and s_4 commute with s_1 and s_2 , we have

$$A_4 s_4^\alpha w_0^\beta s_4^\alpha A_4 \subset \sum_{x \in \mathcal{A}} A_4 s_4^\alpha w_0^\beta s_4^\alpha A_3 x \subset \sum_{x \in \mathcal{A}} A_4 s_4^\alpha w_0^\beta s_4^\alpha x.$$

If moreover $\alpha = \beta$, $s_4^\alpha w_0^\alpha s_4^\alpha = (s_4 s_3 s_2 s_1^2 s_2 s_3 s_4)^\alpha$ commutes with $\langle s_1, s_2, s_3 \rangle = A_4$, hence $A_4 s_4^\alpha w_0^\alpha s_4^\alpha A_4 = A_4 s_4^\alpha w_0^\alpha s_4^\alpha$, and this concludes the proof. □

From this one can conclude the following.

Theorem 6.26.

- (1) $A_5 = A_5^{(3)}$ is generated as a A_4 -module by 240 elements.
- (2) $A_5 = A_5^{(3)}$ is generated as a R -module by 155,520 elements.

Proof. By lemma 6.22, $A_5^{(1)}$ is generated as an A_4 -module by $1 + 2 \times 27 = 55$ elements, $A_5^{(1\frac{1}{4})}$ by $A_5^{(1)}$ and $|\mathcal{B}| = 72$ elements, $A_5^{(1\frac{1}{2})}$ after lemma 6.24 by $A_5^{(1\frac{1}{4})}$ and $2 \times |\mathcal{A}'| = 2 \times 27 = 54$ elements, $A_5^{(2)}$ by $A_5^{(1\frac{1}{2})}$ and $2 + 2 \times |\mathcal{A}| = 56$ elements (lemma 6.25), and $A_5^{(3)}$ by $A_5^{(2)}$ and 3 elements. It follows that A_5 is A_4 -generated by $55 + 72 + 54 + 56 + 3 = 240$ elements, which proves (1). Since A_4 is R -generated by 648 elements, we get that A_5 is R -generated by $240 \times 648 = 155,520$ elements, which proves (2). \square

7. PROOF OF LEMMA 6.16

For the sake of concision we denote $V_0 = A_5^{(2)}$ and $V^+ = A_5^{(2)} + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 = V_0 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4$. We will prove that $X \in A_4^\times s_4 w^- s_4 w^- s_4 A_4^\times + V^+$, starting from $X = \delta^3 = s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4$ to $X = \delta^3 = s_4 w^- s_4 w^- s_4$ (for which the statement is trivial) through a sequence of reductions of the type $X \rightarrow X'$ where $X' \in A_4^\times X A_4^\times + V^+$

7.1. Reduction to $s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4$. Using $s_4^2 \in R^\times s_4^{-1} + R s_4 + R$ we get $s_4 w_0 s_4^2 w_0 s_4^2 w_0 s_4 \in R^\times s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4^2 w_0 s_4 w_0 s_4 + R s_4 w_0 s_4^2 w_0^2 s_4$. The fact that $s_4 w_0 s_4^2 w_0 s_4 w_0 s_4, s_4 w_0 s_4^2 w_0^2 s_4$ belongs to V^+ is proved in the following lemma 7.1

Lemma 7.1.

- (1) $s_4 w_0 s_4^2 w_0^2 s_4 \in V^+$.
- (2) $s_4 w_0 s_4^2 w_0 s_4 w_0 s_4 \in V^+$
- (3) $s_4 w_0^2 s_4^{-1} w_0 s_4 \in V^+$
- (4) $s_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4 \in V_0$

Proof. We prove (1). By lemma 4.9 we have $s_4 w_0 s_4^2 w_0^2 s_4 \in A_3^\times s_4 w_0 s_4^2 w_0^{-1} s_4 + A_3 s_4 w_0 s_4^2 w_0 s_4 + A_3 s_4 w_0 s_4^3$. Clearly $s_4 w_0 s_4^3 \in V_0$, hence, expanding s_4^2 ,

$$s_4 w_0 s_4^2 w_0^2 s_4 \in A_3 s_4 w_0 s_4^{-1} w_0^{-1} s_4 + A_3 s_4 w_0 s_4 w_0^{-1} s_4 + A_3 s_4 w_0 s_4^{-1} w_0 s_4 + A_3 s_4 w_0 s_4 w_0 s_4 + V_0.$$

We already know $s_4 w_0 s_4 w_0 s_4 \in V_0$ by lemma 6.11 (6). Moreover $s_4 w_0 s_4^{-1} w_0^{-1} s_4 \in A_3 s_4 w^+ s_4^{-1} w^- s_4 + V_0 \subset V_0$ and $s_4 w_0 s_4 w_0^{-1} s_4 \in A_3 s_4 w^+ s_4 w^- s_4 + V_0 \subset V_0$ by lemma 6.12 (4), and finally

$$s_4 w_0 s_4^{-1} w_0^{-1} s_4 \in A_3 s_4 w^+ s_4^{-1} w^+ s_4 + V_0 \subset V_+.$$

We now prove (2). We have

$$s_4 w_0 s_4^2 (w_0 s_4 w_0 s_4) = (w_0 s_4 w_0 s_4) s_4 w_0 s_4^2 \subset A_4 s_4 w_0 s_4^2 w_0 s_4^2,$$

as $w_0 s_4 w_0 s_4 = c_5 c_3^{-1}$ commutes with s_4 and w_0 . The term $s_4 w_0 s_4^2 w_0 s_4^2$ is a linear combinations of terms of the form $s_4 w_0 s_4^\alpha w_0 s_4^\beta$ for $\alpha, \beta \in \{0, -1, 1\}$, and we have (lemma 6.11 (6) and (7)) $s_4 w_0 s_4^\alpha w_0 s_4^\beta \in A_3 s_4 w^+ s_4^\alpha w^+ s_4^\beta + V_0 \subset V_0$, unless $(\alpha, \beta) = (-1, 1)$, in which case $s_4 w_0 s_4^{-1} w_0 s_4 \in A_3 s_4 w^+ s_4^{-1} w^+ s_4 + V_0 \subset V^+$. For proving (3) we use that $s_4 w_0^2 s_4^{-1} w_0 s_3 \in V_0 + A_3 \sum_{\alpha \in \{\pm\}} s_4 w^\alpha s_4^{-1} w^+ s_4$, and that $s_4 w^- s_4^{-1} w^+ s_4 \in V_0$ by lemma 6.11 (5). We prove (4). We have $(s_4 w_0 s_4 w_0) s_4^{-1} w_0 s_4 = s_4^{-1} w_0 s_4 (s_4 w_0 s_4 w_0) \in s_4^{-1} w_0 s_4^2 w_0 s_4 A_4$ and $s_4^{-1} w_0 s_4^2 w_0 s_4 \in R s_4^{-1} w_0^2 s_4 + R s_4^{-1} w_0 s_4^{-1} w_0 s_4 + R s_4^{-1} w_0 s_4 w_0 s_4$. Now, $s_4^{-1} w_0^2 s_4 \in V_0$, $s_4^{-1} (w_0 s_4 w_0 s_4) = (w_0 s_4 w_0 s_4) s_4^{-1} \in V_0$, and $s_4^{-1} w_0 s_4^{-1} w_0 s_4 \in A_3 s_4^{-1} w^+ s_4^{-1} w^+ s_4 + V_0$. Since $\Phi(s_4^{-1} w^+ s_4^{-1} w^+ s_4) \in V_0$ by lemma 6.11 (8) we get (4). \square

7.2. Reduction to $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$. Using $s_4^2 \in R^\times s_4^{-1} + R s_4 + R$ we get $s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4 \in R^\times s_4 w_0 s_4^{-1} w_0 s_4^{-1} w_0 s_4 + R s_4 w_0^2 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4$. The fact that $R s_4 w_0^2 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4 \subset V^+$ has been proved in lemma 7.1. Finally, $s_4 w_0 s_4^{-1} w_0 s_4^{-1} w_0 s_4 = s_3^{-1} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ is easily checked to hold in the braid group B_5 .

Before going further, we first need to establish several lemmas.

Lemma 7.2.

- (1) For all $\alpha, \beta \in \mathbf{Z}$, $s_4 w_0 s_4^\alpha s_3^\beta w_0 s_4 \in V^+$.
- (2) For all $\alpha, \beta \in \mathbf{Z}$, $s_4 w_0 s_3^\beta s_4^\alpha w_0 s_4 \in V^+$.

Proof. Since $s_4 w_0 s_4^\alpha s_3^\beta w_0 s_4 = s_4 w_0 s_4^\alpha s_3^{\beta+1} s_2 s_1^2 s_2 s_3 s_4$, we need to consider $s_4 w_0 s_4^\alpha s_3^\beta s_2 s_1^2 s_2 s_3 s_4$, where we can assume $\alpha, \beta \in \{1, -1\}$, the cases $\alpha = 0$ and $\beta = 0$ being obvious by proposition 6.3. If $\beta = 1$ we have $s_4 w_0 s_4^\alpha w_0 s_4 \in A_3 s_4 w^+ s_4^\alpha w^+ s_4 + V_0 \subset V^+$, so we can assume $\beta = -1$. Expanding s_1^2 we get a linear combinations of $s_4 w_0 s_4^\alpha s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4$, $s_4 w_0 s_4^\alpha s_3^{-1} s_2 s_1 s_2 s_3 s_4$ and $s_4 w_0 s_4^\alpha s_3^{-1} s_2^2 s_3 s_4$. We have $s_4 w_0 s_4^\alpha s_3^{-1} s_2^2 s_3 s_4 \in V_0$ by 6.3 and $s_4 w_0 s_4^\alpha s_3^{-1} (s_2 s_1 s_2) s_3 s_4 = s_4 w_0 s_4^\alpha s_3^{-1} s_1 s_2 s_1 s_3 s_4 = s_4 w_0 s_4^\alpha s_1 s_3^{-1} s_2 s_3 s_4 s_1 \in V_0$ by proposition 6.3. There remains to consider $s_4 w_0 s_4^\alpha s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4 = s_4 w_0 s_4^\alpha s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1} s_4 = s_2 s_1 s_4 w_0 s_4^\alpha (s_3 s_2^{-1} s_3) s_4 s_1^{-1} s_2^{-1} \in V_0$ by proposition 6.3. This concludes the proof of (1). Then (2) is an immediate consequence of (1) by application of $\Phi \circ \Psi$. □

Lemma 7.3.

- (1) $s_4 w_0 s_4^{-1} s_3 s_4^{-1} w_0 s_4 \in V^+$
- (2) $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 w_0 s_4^{-1} w_0 s_4 \in V^+$

Proof. By using braid relations one gets $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 w_0 s_4^{-1} w_0 s_4 = s_3 (s_2 s_1^2 s_2) s_4 w_0 s_4^{-1} s_3 s_4^{-1} w_0 s_4$, hence (2) reduces to (1). We now prove (1). Expanding s_4^{-1} as a linear combination of s_4^2 , s_4 and 1, we get that $s_4 w_0 (s_4^{-1}) s_3 s_4^{-1} w_0 s_4$ is a linear combination of $s_4 w_0 s_3 s_4^{-1} w_0 s_4$, $s_4 w_0 s_4 s_3 s_4^{-1} w_0 s_4$ and $s_4 w_0 s_4^2 s_3 s_4^{-1} w_0 s_4$. We have $s_4 w_0 s_3 s_4^{-1} w_0 s_4 \in V_0$ by lemma 7.2 (2), $(s_4 w_0 s_4) s_3 s_4^{-1} w_0 s_4 = s_3 (s_4 w_0 s_4) s_4^{-1} w_0 s_4 = s_3 s_4 w_0^2 s_4 \in V_0$ because $s_4 w_0 s_4 = c_5 c_4^{-1}$ commutes with B_4 and in particular s_3 , and similarly $s_4 w_0 s_4^2 s_3 s_4^{-1} w_0 s_4 = s_4 w_0 s_4 (s_4 s_3 s_4^{-1}) w_0 s_4 = (s_4 w_0 s_4) s_3^{-1} s_4 s_3 w_0 s_4 = s_3^{-1} (s_4 w_0 s_4) s_4 s_3 w_0 s_4 \in A_4 s_4 w_0 s_4^2 s_3 w_0 s_4 \in V^+$ by lemma 7.2 (2). □

Lemma 7.4.

- (1) For all β , $s_4 A_4 s_4 s_3^\beta w_0 s_4 \subset V^+$
- (2) $s_4 s_3 s_2^{-1} s_3 u_1 u_2 s_4 u_3 w_0 s_4 \subset V^+$
- (3) $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \in V^+$
- (4) $u_4 u_3 u_2 u_3 u_4 A_4 s_4 \subset V^+$; moreover $s_4^\alpha u_3 u_2 u_3 s_4^\beta A_4 s_4 \subset V_0$ when $\alpha, \beta \in \{-1, 1\}$ with $(\alpha, \beta) \neq (1, 1)$
- (5) $u_4 U_0 u_4 A_4 s_4 \subset V^+$
- (6) $u_4 s_3^\pm A_3 s_3^\mp u_4 A_4 s_4 \subset V^+$

Proof. From theorem 4.1 one easily deduces $A_4 = A_3 u_3 A_3 + A_3 s_3 s_2^{-1} s_3 A_3 + A_3 w_0 + A_3 w_0^{-1}$. Then $s_4 (A_3 u_3 A_3) s_4 s_3^\beta w_0 s_4 = A_3 s_4 u_3 s_4 A_3 s_3^\beta w_0 s_4 \subset V_0$ by lemma 6.11 (2) ; $s_4 (A_3 w_0) s_4 s_3^\beta w_0 s_4 = A_3 s_4 w_0 s_4 s_3^\beta w_0 s_4 \subset V^+$ by lemma 7.2 (2) ; $s_4 (A_3 w_0^{-1}) s_4 s_3^\beta w_0 s_4 = A_3 s_4 w_0^{-1} s_4 s_3^\beta w_0 s_4$ and $s_4 w_0^{-1} s_4 s_3^\beta w_0 s_4$ is a linear combination of $s_4 w_0^{-1} s_4 s_3^{\beta'} s_2 s_1^2 s_2 s_3 s_4$ for $\beta' \in \{0, 1, -1\}$. For $\beta' = 1$, $s_4 w_0^{-1} s_4 s_3 s_2 s_1^2 s_2 s_3 s_4 = s_4 w_0^{-1} s_4 w_0 s_4 \in A_3 s_4 w^- s_4 w^+ s_4 + V_0 \subset V_0$ by lemma 6.11 (5). For $\beta' = -1$, $s_4 w_0^{-1} s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 \in V_0$ by lemma 6.8 (4), and the case $\beta = 0$ also lies in V_0 by proposition 6.3. It then remains to prove $s_4 A_3 s_3 s_2^{-1} s_3 A_3 s_4 s_3^\beta w_0 s_4 \subset V^+$, that is $s_4 s_3 s_2^{-1} s_3 A_3 s_4 s_3^\beta w_0 s_4 \subset V^+$. We use $A_3 \subset s_2^{-1} s_1 s_2^{-1} u_1 + u_1 u_2 u_1$ to get $s_4 s_3 s_2^{-1} s_3 A_3 s_4 s_3^\beta w_0 s_4 \subset s_4 s_3 s_2^{-1} s_3 s_2^{-1} s_1 s_2^{-1} u_1 s_4 s_3^\beta w_0 s_4 + s_4 s_3 s_2^{-1} s_3 u_1 u_2 u_1 s_4 s_3^\beta w_0 s_4$. Now, for $s_4 s_3 s_2^{-1} s_3 u_1 u_2 u_1 s_4 s_3^\beta w_0 s_4 = s_4 s_3 s_2^{-1} s_3 u_1 u_2 s_4 s_3^\beta w_0 s_4 u_1$ we are reduced to proving (2), while the expression $s_4 (s_3 s_2^{-1} s_3 s_2^{-1}) s_1 s_2^{-1} u_1 s_4 s_3^\beta w_0 s_4 = s_4 (s_3 s_2^{-1} s_3 s_2^{-1}) s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 u_1$ is

$$s_4 s_2^{-1} s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 u_1 + s_4 u_2 u_3 s_1 s_2^{-1} u_1 s_4 s_3^\beta w_0 s_4 + s_4 u_3 u_2 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 u_1$$

by lemma 3.6. We have $s_4 u_2 u_3 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 + s_4 u_3 u_2 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 \subset V_0$ by proposition 6.3, while $s_4 s_2^{-1} s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4 = s_2^{-1} s_4 s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4$ $s_4 s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^\beta w_0 s_4$ is a linear combination of $s_4 s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^{\beta'} s_2 s_1^2 s_2 s_3 s_4$ for $\beta' \in \{0, 1, -1\}$. Now

$$\begin{aligned} s_4 s_3 s_2^{-1} s_3 s_1 s_2^{-1} s_4 s_3^{\beta'} s_2 s_1^2 s_2 s_3 s_4 &= s_4 s_3 s_2^{-1} s_3 s_1 s_4 (s_2^{-1} s_3^{\beta'} s_2) s_1^2 s_2 s_3 s_4 \\ &= s_4 s_3 s_2^{-1} s_3 s_1 s_4 s_3 s_2^{\beta'} s_3^{-1} s_1^2 s_2 s_3 s_4 = s_4 s_3 s_2^{-1} s_3 s_1 s_4 s_3 s_2^{\beta'} s_1^2 (s_3^{-1} s_2 s_3) s_4 \\ &= s_4 s_3 s_2^{-1} s_3 s_1 s_4 s_3 s_2^{\beta'} s_1^2 s_2 s_3 s_2^{-1} s_4 = s_4 s_3 s_2^{-1} s_3 s_1 s_4 s_3 s_2^{\beta'} s_1^2 s_2 s_3 s_4 s_2^{-1} \in V_0 \end{aligned}$$

by proposition 6.3. This proves (1). For proving (2), we can reduce to an expression of the form $s_4 s_3 s_2^{-1} s_3 s_1^\alpha s_2^\beta s_4 s_3^\gamma s_2 s_1^2 s_2 s_3 s_4$ (with $\alpha, \beta, \gamma \in \{0, 1, -1\}$). Using only braid relations, one gets

$$\begin{aligned} s_4 s_3 s_2^{-1} s_3 s_1^\alpha s_2^\beta s_4 s_3^\gamma s_2 s_1^2 s_2 s_3 s_4 &= s_4 s_3 s_2^{-1} s_3 s_1^\alpha s_2^\beta s_4 s_3^\gamma s_2 s_1^2 s_2 (s_3 s_4 s_3^{-1}) s_3 \\ &= s_4 s_3 s_2^{-1} s_3 s_1^\alpha s_2^\beta s_4 s_3^\gamma s_2 s_1^2 s_2 s_4^{-1} s_3 s_4 s_3 = s_4 s_3 s_2^{-1} s_3 s_1^\alpha s_2^\beta (s_4 s_3^\gamma s_4^{-1}) s_2 s_1^2 s_2 s_3 s_4 s_3 \\ &= s_4 s_3 s_2^{-1} s_1^\alpha (s_3 s_2^\beta s_3^{-1}) s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_4 s_3 = s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_2 s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_4 s_3 \\ &= s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_4^\gamma (s_2 s_3 s_2) s_1^2 s_2 s_3 s_4 s_3 = s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_4^\gamma s_3 s_2 s_3 s_1^2 s_2 s_3 s_4 s_3 \\ &= s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_4^\gamma s_3 s_2 s_1^2 (s_3 s_2 s_3) s_4 s_3 = s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_2 s_4 s_3 \\ &= s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3^\beta s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_4 \cdot s_2 s_3 \end{aligned}$$

which belongs to V_0 by lemma 6.8 (3) as soon as $\beta = -1$. If $\beta = 0$, it is equal to

$$s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_4 \cdot s_2 s_3 = s_4 s_3 s_4^\gamma s_2^{-1} s_1^\alpha s_2^{-1} s_3 s_2 s_1^2 s_2 s_3 s_4 \cdot s_2 s_3 \in V_0$$

by lemma 6.11 (2). Otherwise, considering all possibilities for α and applying lemmas 6.7 and 6.8 it lies inside $V_0 + s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4^\gamma s_3 s_2 s_1^2 s_2 s_3 s_4 A_4 \subset s_4 w^+ s_4^\gamma w^+ s_4 A_4 + V_0$. In cases $\gamma \in \{-1, 0\}$ this clearly belongs to V^+ , while $s_4 w^+ s_4 w^+ s_4 \in V_0$ by lemma 6.11 (6).

We now prove (3). We have

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &= s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} (s_2 s_1^2 s_2) (s_4 s_3 s_4^{-1}) w_0 s_4 \\ &= s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3 w_0 s_4 \in s_4 A_4 s_4 s_3 w_0 s_4 \subset V^+ \end{aligned}$$

by (1).

We prove (4), considering an expression of the form $s_4^\alpha u_3 u_2 u_3 s_4^\beta A_4 s_4$ for $\alpha, \beta \in \{-1, 1\}$, the case $\alpha = 0$ or $\beta = 0$ being obvious. We use the decomposition $A_4 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3 + u_3 u_2 u_1 u_2 u_3 A_3$. We have

$$\begin{aligned} s_4^\alpha u_3 u_2 u_3 s_4^\beta A_3 u_3 A_3 s_4 &= s_4^\alpha u_3 u_2 u_3 s_4^\beta A_3 u_3 s_4 A_3 \\ &= s_4^\alpha u_3 u_2 u_3 s_4^\beta u_2 u_1 u_2 u_1 u_3 s_4 A_3 = s_4^\alpha u_3 u_2 u_3 s_4^\beta u_2 u_1 u_2 u_3 s_4 u_1 A_3 \subset V_0 \end{aligned}$$

by proposition 6.3, and $s_4^\alpha u_3 u_2 u_3 s_4^\beta u_3 u_2 u_1 u_2 u_3 A_3 s_4 = s_4^\alpha u_3 u_2 u_3 s_4^\beta u_3 u_2 u_1 u_2 u_3 s_4 A_3 \subset V_0$ by proposition 6.3. There remains to consider

$$\begin{aligned} s_4^\alpha u_3 u_2 u_3 s_4^\beta A_3 u_3 u_2 u_3 A_3 s_4 &= s_4^\alpha u_3 u_2 u_3 s_4^\beta A_3 u_3 u_2 u_3 s_4 A_3 \\ &= s_4^\alpha u_3 u_2 u_3 s_4^\beta u_1 u_2 u_1 (u_2 u_3 u_2 u_3) s_4 A_3 = s_4^\alpha u_3 u_2 u_3 s_4^\beta u_1 u_2 u_1 u_3 u_2 u_3 s_4 A_3 \\ &= s_4^\alpha u_3 u_2 u_3 s_4^\beta u_1 u_2 u_1 u_3 u_2 u_3 s_4 u_2 A_3 \subset V_0 \end{aligned}$$

by proposition 6.3, unless $\alpha = \beta = 1$. In that case the proof of lemma 6.4, lemma 6.12 (1) and lemma 6.15 (2) together yield $s_4 u_3 u_2 u_3 s_4 u_1 u_2 u_1 u_3 u_2 u_3 s_4 \in V^+$.

Now (5) is a consequence of (4) and proposition 6.3, as $U_0 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3$ and

$$\begin{aligned} u_4 U_0 u_4 A_4 s_4 &\subset u_4 A_3 u_3 A_3 u_4 A_4 s_4 + u_4 A_3 u_3 u_2 u_3 A_3 u_4 A_4 s_4 \\ &= A_3 u_4 u_3 u_4 A_4 s_4 + u_4 A_3 u_3 u_2 u_3 u_4 A_4 s_4 \end{aligned}$$

and both terms belong to V^+ , by lemma 6.11 (2) and by (4). Then (6) is an immediate consequence of (5) and lemma 4.6 (3). \square

7.3. Reduction to $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$. Expanding s_3^2 , we get

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &\in R^\times s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &\quad + R s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &\quad + R s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4. \end{aligned}$$

We have

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &\in V^+ \quad \text{by lemma 7.4 (3)} \\ s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &\in V^+ \quad \text{by lemma 7.3 (2)}. \end{aligned}$$

Lemma 7.5.

- (1) For all $\alpha \in \mathbf{Z}$ $s_4 s_3^{-1} s_2^\alpha s_3^{-1} s_4^{-1} s_3 w_0 s_4 \in V^+$
- (2) $s_4 s_3^{-1} s_2^2 s_3^{-1} s_4^2 s_3 w_0 s_4 \in V^+$
- (3) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_2^2 s_3 s_4^{-1} w_0 s_4 \in V^+$
- (4) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1 s_2 s_3 s_4^{-1} w_0 s_4 \in V^+$

Proof. We prove (1). We get

$$\begin{aligned} s_4 s_3^{-1} s_2^\alpha (s_3^{-1} s_4^{-1} s_3) w_0 s_4 &= s_4 s_3^{-1} s_2^\alpha s_4 (s_3^{-1} s_4^{-1} s_3) s_2 s_1^2 s_2 s_3 s_4 \\ &= s_4 s_3^{-1} s_2^\alpha s_4 s_4 s_3^{-1} s_4^{-1} s_2 s_1^2 s_2 s_3 s_4 = s_4 s_3^{-1} s_2^\alpha s_4 s_3^{-1} s_2 s_1^2 s_2 (s_4^{-1} s_3 s_4) \\ &= s_4 s_3^{-1} s_2^\alpha s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} \in V_0 \end{aligned}$$

by lemma 6.11 (2). Part (2) is obtained by expanding s_4^2 and using (1) and lemma 7.4 (1). For (3), we use

$$\begin{aligned} s_4 s_3^{-1} s_2 s_4 (s_3^{-1} s_2^2 s_3) s_4^{-1} w_0 s_4 &= s_4 s_3^{-1} s_2 s_4 s_2 s_3^2 s_2^{-1} s_4^{-1} w_0 s_4 \\ &= s_4 s_3^{-1} s_2^2 (s_4 s_3^2 s_4^{-1}) w_0 s_4 s_2^{-1} = s_4 s_3^{-1} s_2^2 s_3^{-1} s_4^2 s_3 w_0 s_4 s_2^{-1} \in V^+ \end{aligned}$$

because of (2). For (4) we use $s_4 s_3^{-1} s_2 s_4 s_3^{-1} (s_2 s_1 s_2) s_3 s_4^{-1} w_0 s_4 = s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_1 s_2 s_1 s_3 s_4^{-1} w_0 s_4 = s_4 s_3^{-1} s_2 s_4 s_1 (s_3^{-1} s_2 s_3) s_4^{-1} w_0 s_4 s_1 = s_4 s_3^{-1} s_2 s_4 s_1 s_2 s_3 s_2^{-1} s_4^{-1} w_0 s_4 s_1 = s_4 s_3^{-1} s_2 s_1 s_2 s_4 s_3 s_4^{-1} w_0 s_4 s_2^{-1} s_1 = s_4 s_3^{-1} s_2 s_1 s_2 s_3^{-1} s_4 s_3 w_0 s_4 s_2^{-1} s_1 \in V^+$ by lemma 7.4 (1). □

Lemma 7.6.

- (1) $s_4 s_3^{-1} A_3 s_4 u_3 s_4^{-1} w_0 s_4 \subset V^+$
- (2) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} u_1 u_2 u_1 s_3 s_4^{-1} w_0 s_4 \subset V^+$
- (3) $s_4 s_3^{-1} s_2 s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4^{-1} w_0 s_4 \in V^+$

Proof. We consider $s_4 s_3^{-1} A_3 s_4 s_3^\alpha s_4^{-1} w_0 s_4$ for $\alpha \in \{-1, 0, 1\}$. When $\alpha = 0$ this expression clearly belongs to V_0 , when $\alpha = 1$ we get $s_4 s_3^{-1} A_3 (s_4 s_3 s_4^{-1}) w_0 s_4 = s_4 s_3^{-1} A_3 s_3^{-1} s_4 s_3 w_0 s_4 \subset V^+$ by lemma 7.4 (1). When $\alpha = -1$, we get $s_4 s_3^{-1} A_3 s_4 (s_3^{-1} s_4^{-1} s_3) s_2 s_1^2 s_2 s_3 s_4 = s_4 s_3^{-1} A_3 s_4 s_4 s_3^{-1} s_4^{-1} s_2 s_1^2 s_2 s_3 s_4 = s_4 s_3^{-1} A_3 s_4 s_3^{-1} s_2 s_1^2 s_2 (s_4^{-1} s_3 s_4) = s_4 s_3^{-1} s_4^2 A_3 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} \subset V_0$ by lemma 6.11 (1). This proves (1). We consider now $s_4 s_3^{-1} s_2 s_4 s_3^{-1} u_1 u_2 u_1 s_3 s_4^{-1} w_0 s_4 = s_4 s_3^{-1} s_2 u_1 s_4 (s_3^{-1} u_2 s_3) s_4^{-1} w_0 s_4 u_1 = s_4 s_3^{-1} s_2 u_1 s_4 s_2 u_3 s_2^{-1} s_4^{-1} w_0 s_4 u_1 = s_4 s_3^{-1} s_2 u_1 s_2 (s_4 u_3 s_4^{-1}) w_0 s_4 s_2^{-1} u_1 = s_4 s_3^{-1} s_2 u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 s_2^{-1} u_1$ which is a linear combination of the $Y = s_4 s_3^{-1} s_2 u_1 s_2 s_3^{-1} s_4^\alpha s_3 w_0 s_4 s_2^{-1} u_1$ for $\alpha \in \{-1, 0, 1\}$. When $\alpha = 0$ clearly $Y \subset V_0$, when $\alpha = 1$ we have $Y \subset V^+$ by lemma 7.4 (1), and when $\alpha = -1$ we get $s_4 s_3^{-1} s_2 u_1 s_2 (s_3^{-1} s_4^{-1} s_3) w_0 s_4 s_2^{-1} u_1 = s_4 s_3^{-1} s_2 u_1 s_2 s_4 s_3^{-1} s_4^{-1} w_0 s_4 s_2^{-1} u_1 \subset V^+$ by (1). This proves (2). We consider now $s_4 s_3^{-1} s_2 s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4^{-1} w_0 s_4 = s_4 s_3^{-1} s_2 s_4 s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1} s_4^{-1} w_0 s_4 = s_4 s_3^{-1} s_2^2 s_1 s_4 (s_3 s_2^{-1} s_3) s_4^{-1} w_0 s_4 s_1^{-1} s_2^{-1}$. By lemma 2.4 it belongs to

$$s_4 s_3^{-1} s_2^2 s_1 s_4 s_3^{-1} s_2 s_3^{-1} u_2 s_4^{-1} w_0 s_4 s_1^{-1} s_2^{-1} + s_4 s_3^{-1} s_2^2 s_1 s_4 u_2 u_3 u_2 s_4^{-1} w_0 s_4 s_1^{-1} s_2^{-1}.$$

We have

$$s_4 s_3^{-1} s_2^2 s_1 s_4 u_2 u_3 u_2 s_4^{-1} w_0 s_4 s_1^{-1} s_2^{-1} = s_4 s_3^{-1} s_2^2 s_1 u_2 s_4 u_3 s_4^{-1} w_0 s_4 u_2 s_1^{-1} s_2^{-1} \subset V^+$$

by (1), and

$$\begin{aligned} s_4 s_3^{-1} s_2^2 s_1 s_4 s_3^{-1} s_2 s_3^{-1} u_2 s_4^{-1} w_0 s_4 s_1^{-1} s_2^{-1} &= s_4 s_3^{-1} s_2^2 s_1 s_4 s_3^{-1} s_2 (s_3^{-1} s_4^{-1} s_3) s_2 s_1^2 s_2 s_3 s_4 u_2 s_1^{-1} s_2^{-1} \\ &= s_4 s_3^{-1} s_2^2 s_1 s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_4^{-1} s_2 s_1^2 s_2 s_3 s_4 u_2 s_1^{-1} s_2^{-1} \\ &= s_4 s_3^{-1} s_2^2 s_1 s_4 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 (s_4^{-1} s_3 s_4) u_2 s_1^{-1} s_2^{-1} \\ &= (s_4 s_3^{-1} s_4) s_2^2 s_1 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} u_2 s_1^{-1} s_2^{-1} \end{aligned}$$

belongs to

$$u_3 s_4^{-1} s_3 s_4^{-1} s_2^2 s_1 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} u_2 s_1^{-1} s_2^{-1} + u_3 u_4 u_3 s_2^2 s_1 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} u_2 s_1^{-1} s_2^{-1}.$$

We have

$$\begin{aligned} s_4^{-1} s_3 s_4^{-1} s_2^2 s_1 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 &= s_4^{-1} s_3 s_2^2 s_1 (s_4^{-1} s_3^{-1} s_4) s_2 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 \\ &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_4^{-1} s_3^{-1} s_2 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 = s_4^{-1} s_3 s_2^2 s_3 s_4^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 \in u_4 u_3 u_2 u_3 u_4 A_4 s_4 \end{aligned}$$

and also $u_4 u_3 s_2^2 s_1 s_3^{-1} s_2 s_4 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} u_2 s_1^{-1} s_2^{-1} \subset u_4 u_3 u_2 u_3 u_4 A_4 s_4 A_4$. The conclusion follows from 7.4 (4). \square

7.4. Reduction to $s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$. Expanding s_1^2 , we get

$$\begin{aligned} s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &\in R^\times s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &\quad + R s_4 s_3 (s_2 s_1 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &\quad + R s_4 s_3 s_2^2 s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \end{aligned}$$

Since

$$\begin{aligned} s_4 s_3 (s_2 s_1 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &= s_4 s_3 s_1 s_2 s_1 s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &= s_1 s_4 s_3 s_2 s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 s_1 \end{aligned}$$

(as s_1 commutes with $s_2 s_1^2 s_2 = c_3 c_2^{-1}$), the latter two terms belong to

$$\begin{aligned} A_2 s_4 (s_3 u_2 s_3^{-1}) s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2 &= A_2 s_4 s_2^{-1} u_3 s_2 s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2 \\ &\subset A_3 s_4 u_3 s_4 s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 A_2. \end{aligned}$$

We thus only need to prove that the $s_4 s_3^\alpha s_4 s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$ belong to V^+ for $\alpha \in \{-1, 0, 1\}$. When $\alpha = 0$ this is a consequence of lemma 7.4 (6); when $\alpha = 1$ we get

$$\begin{aligned} (s_4 s_3 s_4) s_2 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 &= s_3 s_4 (s_3 s_2 s_3^{-1}) (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \\ &= s_3 s_4 s_2^{-1} s_3 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4. \end{aligned}$$

Since $s_2^2 s_1^2 s_2 \in u_1 s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ the conclusion follows from proposition 6.3. When $\alpha = -1$, expanding s_1^2 we only need to consider the $s_4 s_3^{-1} s_4 s_2 s_3^{-1} s_2 s_1^\beta s_2 s_3 s_4^{-1} w_0 s_4$ for $\beta \in \{-1, 0, 1\}$. The case $\beta = -1$ is a consequence of lemma 7.6 (3), while the other two cases follow from lemma 7.5 (3) and (4).

Lemma 7.7.

- (1) $s_4 u_3 s_2 u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 \subset V^+$
- (2) $s_4 s_3 s_2 s_1^{-1} u_3 u_2 u_4 s_3 w_0 s_4 \subset V_0$
- (3) $s_4 s_3 s_2 s_1^{-1} u_2 u_3 u_4 s_3 w_0 s_4 \subset V^+$

Proof. For proving (1), we consider the expression $s_4 s_3^\alpha s_2 s_1^\beta s_2 s_3^\gamma u_4 s_3 w_0 s_4$ for $\alpha, \beta, \gamma \in \{-1, 0, 1\}$. If one of these is 0, it lies in V_0 by proposition 6.3. If $\beta = 1$, using $s_2 s_1 s_2 = s_1 s_2 s_1$ we get the same conclusion, so we can assume $\beta = -1$. By expanding if necessary s_3^2 , we are then reduced to considering expressions of the form $s_4 s_3^\alpha s_2 s_1^\beta s_2 s_3^\gamma u_4 s_3^\delta s_2 s_1^2 s_2 s_3 s_4$ for $\delta \in \{0, 1, -1\}$, the case $\delta = 0$ being again trivial. We then get the conclusion from lemmas 6.8 and 6.12 (6).

We now prove (2). We have

$$\begin{aligned} s_4 s_3 s_2 s_1^{-1} u_3 u_2 u_4 s_3 w_0 s_4 &= s_4 (s_3 s_2 u_3) s_1^{-1} u_2 u_4 s_3 w_0 s_4 \\ \subset s_4 u_2 s_3 s_2 s_1^{-1} u_2 u_4 s_3 w_0 s_4 &= u_2 s_4 s_3 s_2 s_1^{-1} u_2 u_4 s_3 w_0 s_4 \subset V_0 \end{aligned}$$

by proposition 6.3. Finally, (3) is similar to (1): considering $s_4 s_3 s_2 s_1^{-1} s_2^\alpha s_3^\beta s_4^\gamma s_3^\delta s_2 s_1^2 s_2 s_3 s_4$, if one of the exponents is zero we get trivially the conclusion by proposition 6.3; if $\alpha = -1$ it lies inside V_0 by $s_2 s_1^{-1} s_2^{-1} = s_1^{-1} s_2^{-1} s_1$ and proposition 6.3, so we can assume $\alpha = 1$. By studying separately the cases $\beta = -1$ and $\beta = 1$ one easily gets the conclusion from lemmas 6.8, 6.11 and 6.12. \square

7.5. Reduction to $s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} w_0 s_4$. Using $s_2 s_1^2 s_2 \in s_2^{-1} s_1 s_2^{-1} u_1^\times + u_1 u_2 u_1$ we get

$$s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4 \in R^\times s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} w_0 s_4 u_1^\times + R s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} u_1 u_2 u_1 s_3 s_4^{-1} w_0 s_4.$$

We have

$$\begin{aligned} s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} u_1 u_2 u_1 s_3 s_4^{-1} w_0 s_4 &= s_4 s_3 (s_2 s_1^{-1} s_2) u_1 s_3^{-1} s_4 (s_3^{-1} u_2 s_3) s_4^{-1} w_0 s_4 u_1 \\ &= s_4 s_3 (s_2 s_1^{-1} s_2) u_1 s_3^{-1} s_4 s_2 u_3 s_2^{-1} s_4^{-1} w_0 s_4 u_1 = s_4 s_3 (s_2 s_1^{-1} s_2) u_1 s_3^{-1} s_2 (s_4 u_3 s_4^{-1}) w_0 s_4 s_2^{-1} u_1 \\ &= s_4 s_3 (s_2 s_1^{-1} s_2) u_1 s_3^{-1} s_2 s_3^{-1} u_4 s_3 w_0 s_4 s_2^{-1} u_1. \end{aligned}$$

Using $(s_2 s_1^{-1} s_2) u_1 \in u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$ we get that $s_4 s_3 (s_2 s_1^{-1} s_2) u_1 s_3^{-1} s_2 s_3^{-1} u_4 s_3 w_0 s_4$ belongs to

$$u_1 s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_2 s_3^{-1} u_4 s_3 w_0 s_4 + s_4 s_3 u_1 u_2 u_1 s_3^{-1} s_2 s_3^{-1} u_4 s_3 w_0 s_4.$$

Now

$$\begin{aligned} s_4 s_3 u_1 u_2 u_1 s_3^{-1} s_2 s_3^{-1} u_4 s_3 w_0 s_4 &= u_1 s_4 (s_3 u_2 s_3^{-1}) u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 \\ &= u_1 s_4 s_2^{-1} u_3 s_2 u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = u_1 s_2^{-1} s_4 u_3 s_2 u_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 \subset V^+ \end{aligned}$$

by lemma 7.7 (1), and $s_4 s_3 s_2 s_1^{-1} (s_2 s_3^{-1} s_2 s_3^{-1}) u_4 s_3 w_0 s_4$ belong to

$$s_4 s_3 s_2 s_1^{-1} s_3^{-1} s_2 s_3^{-1} s_2 u_4 s_3 w_0 s_4 + s_4 s_3 s_2 s_1^{-1} u_2 u_3 u_4 s_3 w_0 s_4 + s_4 s_3 s_2 s_1^{-1} u_3 u_2 u_4 s_3 w_0 s_4$$

by lemma 3.6. The latter two terms belong to V^+ by lemma 7.7 (1) and (2), and

$$\begin{aligned} s_4 s_3 s_2 s_1^{-1} s_3^{-1} s_2 s_3^{-1} s_2 u_4 s_3 w_0 s_4 &= s_4 (s_3 s_2 s_3^{-1}) s_1^{-1} s_2 s_3^{-1} s_2 u_4 s_3 w_0 s_4 \\ &= s_4 s_2^{-1} s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_2 u_4 s_3 w_0 s_4 = s_2^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} u_4 s_2 s_3 w_0 s_4 \subset V^+ \end{aligned}$$

by lemma 7.4 (6).

Lemma 7.8. $s_4 (s_3 s_2^{-1} s_3) u_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 \subset V^+$

Proof. Using braid relations one gets

$$\begin{aligned} s_4 (s_3 s_2^{-1} s_3) u_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 &= s_4 s_3 s_2^{-1} u_1 (s_3 s_4 s_3^{-1}) s_2 s_3^{-1} s_4^{-1} w_0 s_4 \\ &= s_4 s_3 s_2^{-1} u_1 s_4^{-1} s_3 s_4 s_2 s_3^{-1} s_4^{-1} w_0 s_4 = (s_4 s_3 s_4^{-1}) s_2^{-1} u_1 s_3 s_2 (s_4 s_3^{-1} s_4^{-1}) w_0 s_4 \\ &= s_3^{-1} s_4 s_3 s_2^{-1} u_1 (s_3 s_2 s_3^{-1}) s_4^{-1} s_3 w_0 s_4 = s_3^{-1} s_4 s_3 s_2^{-1} u_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3 w_0 s_4 \end{aligned}$$

which is a linear combination of $s_3^{-1} s_4 s_3 s_2^{-1} s_1^\alpha s_2^{-1} s_3 s_2 s_4^{-1} s_3 w_0 s_4$ for $\alpha \in \{-1, 0, 1\}$. For $\alpha = -1$ we get

$$\begin{aligned} s_3^{-1} s_4 s_3 (s_2^{-1} s_1^{-1} s_2^{-1}) s_3 s_2 s_4^{-1} s_3 w_0 s_4 &= s_3^{-1} s_4 s_3 s_1^{-1} s_2^{-1} s_1^{-1} s_3 s_2 s_4^{-1} s_3 w_0 s_4 \\ &= s_3^{-1} s_1^{-1} s_4 s_3 s_2^{-1} s_3 s_4^{-1} s_1^{-1} s_2 s_3 w_0 s_4 \in V^+ \end{aligned}$$

by lemma 7.4 (4) ; for $\alpha = 0$ we get $s_3^{-1} s_4 s_3 s_2^{-2} s_3 s_2 s_4^{-1} s_3 w_0 s_4 \in V_0$ by proposition 6.3 ; for $\alpha = 1$ it remains to consider $s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_2 s_4^{-1} s_3 w_0 s_4$. Using $s_2^{-1} s_1 s_2^{-1} \in u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$ we get $s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_2 s_4^{-1} s_3 w_0 s_4 \in u_1 s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_2 s_4^{-1} s_3 w_0 s_4 + u_1 s_4 s_3 u_2 u_1 s_3 s_2 s_4^{-1} s_3 w_0 s_4$. Now $s_4 s_3 u_2 u_1 s_3 s_2 s_4^{-1} s_3 w_0 s_4 = s_4 s_3 u_2 s_3 s_4^{-1} u_1 s_2 s_3 w_0 s_4 \subset V^+$ by lemma 7.4 (4) while

$$\begin{aligned} s_4 s_3 s_2 s_1^{-1} (s_2 s_3 s_2) s_4^{-1} s_3 w_0 s_4 &= s_4 s_3 s_2 s_1^{-1} s_3 s_2 s_3 s_4^{-1} s_3 w_0 s_4 \\ &= s_4 (s_3 s_2 s_3) s_1^{-1} s_2 s_3 s_4^{-1} s_3 w_0 s_4 = s_4 s_2 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 w_0 s_4 \\ &= s_2 s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 w_0 s_4 \end{aligned}$$

lies in V^+ by lemma 6.8. This concludes the proof. \square

Lemma 7.9. $s_4 (s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 \subset V^+$

Proof. We have

$$\begin{aligned} s_4 (s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 &= s_4 (s_3 s_2^{-1} s_3) s_2 s_1 u_2 (s_4 u_3 s_4^{-1}) w_0 s_4 \\ &\subset s_4 s_3 (s_2^{-1} s_3 s_2) s_1 u_2 s_3^{-1} u_4 s_3 w_0 s_4 \\ &= s_4 s_3^2 s_2 s_3^{-1} s_1 u_2 s_3^{-1} u_4 s_3 w_0 s_4, \end{aligned}$$

whose elements are linear combinations of the $s_4 s_3^2 s_2 s_3^{-1} s_1 s_2^\alpha s_3^{-1} u_4 s_3 w_0 s_4$ for $\alpha \in \{0, 1, -1\}$. When $\alpha = 0$, such an element belongs to V_0 by proposition 6.3 ; when $\alpha = -1$, we have $s_4 s_3^2 s_2 s_3^{-1} s_1 s_2^{-1} s_3^{-1} u_4 s_3 w_0 s_4 = s_4 s_3^2 s_2 s_1 (s_3^{-1} s_2^{-1} s_3^{-1}) u_4 s_3 w_0 s_4 = s_4 s_3^2 (s_2 s_1 s_2^{-1}) s_3^{-1} s_2^{-1} u_4 s_3 w_0 s_4 =$

$s_4 s_3^2 s_1^{-1} s_2 s_1 s_3^{-1} s_2^{-1} u_4 s_3 w_0 s_4 = s_1^{-1} s_4 s_3^2 s_2 s_3^{-1} u_4 s_1 s_2^{-1} s_3 w_0 s_4 \in V^+$ by lemma 7.4 (4) ; when $\alpha = 1$, expanding s_3^2 we get a linear combination of $s_4 s_3^\beta s_2 s_3^{-1} s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4$ for $\beta \in \{0, 1, -1\}$. When $\beta = 0$ such an element lies in V_0 by commutation relations and proposition 6.3 ; when $\beta = 1$ we get $s_4(s_3 s_2 s_3^{-1}) s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_4 s_2^{-1} s_3 s_2 s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_4 s_3 (s_2 s_1 s_2) s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_4 s_3 s_1 s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_1 s_4 s_3 s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 \in V_0$ by proposition 6.3 ; when $\beta = -1$ we get that $s_4 s_3^{-1} s_2 s_3^{-1} s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_4 s_3^{-1} s_2 s_1 (s_3^{-1} s_2 s_3^{-1}) u_4 s_3 w_0 s_4$ belongs to

$$s_4 s_3^{-1} s_2 s_1 u_2 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 + s_4 s_3^{-1} s_2 s_1 u_2 u_3 u_2 u_4 s_3 w_0 s_4$$

by lemma 2.4. Now

$$\begin{aligned} s_4 s_3^{-1} (s_2 s_1 u_2) u_3 u_2 u_4 s_3 w_0 s_4 &\subset s_4 s_3^{-1} u_1 s_2 s_1 u_3 u_2 u_4 s_3 w_0 s_4 \\ &\subset u_1 s_4 s_3^{-1} s_2 u_3 u_4 s_1 u_2 s_3 w_0 s_4 \subset V^+ \end{aligned}$$

by lemma 7.4 (4) and

$$\begin{aligned} s_4 s_3^{-1} (s_2 s_1 u_2) s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 &= s_4 s_3^{-1} u_1 s_2 s_1 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \\ &= u_1 s_4 s_3^{-1} s_2 s_1 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 = u_1 s_4 (s_3^{-1} s_2 s_3) s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \\ &= u_1 s_4 s_2 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 = u_1 s_2 s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4. \end{aligned}$$

Since $s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4$ is spanned by the $s_4 w^+ s_4^{\alpha\beta} s_2 s_1^2 s_2 s_3 s_4$ one readily gets

$$s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \subset V^+$$

and the conclusion. \square

Lemma 7.10.

- (1) $s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4 \subset V_0$
- (2) $u_4 u_3 u_2 u_1 s_3^{-1} s_2 s_3^{-1} s_4^{-1} w_0 s_4 \subset V^+$

Proof. We prove (1).

$$\begin{aligned} s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4 &= s_4(s_3 s_2^{-1} s_3) s_2 s_1 (s_4 u_3 s_4^{-1}) w_0 s_4 u_2 \\ &\subset s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 u_2 = s_4 s_3 s_2^{-1} (s_3 s_2 s_3^{-1}) s_1 u_4 s_3 w_0 s_4 u_2 \\ &= s_4 s_3 s_2^{-2} s_3 s_2 s_1 u_4 s_3 w_0 s_4 u_2 \subset s_4 u_3 u_2 u_3 u_2 s_1 u_4 s_3 w_0 s_4 u_2 \\ &= s_4 u_2 u_3 u_2 u_3 s_1 u_4 s_3 w_0 s_4 u_2 = u_2 s_4 u_3 u_2 u_3 s_1 u_4 s_3 w_0 s_4 u_2 \subset V_0 \end{aligned}$$

by proposition 6.3. We now prove (2). We have

$$\begin{aligned} u_4 u_3 u_2 u_1 s_3^{-1} s_2 s_3^{-1} s_4^{-1} w_0 s_4 &= u_4 (u_3 u_2 s_3^{-1}) u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 \\ &\subset u_4 u_2 s_3 s_2^{-1} s_3 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 + u_4 u_2 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4, \end{aligned}$$

and $u_4 u_2 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 = u_2 u_4 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4$ with $u_4 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 \subset V_0 + A_4 s_4 w^- s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} w^+ s_4 A_4 \subset V_0$ by lemma 6.11 (3) and (4). Moreover

$$\begin{aligned} u_4 u_2 s_3 s_2^{-1} s_3 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 &= u_2 u_4 s_3 s_2^{-1} u_1 (s_3 s_2 s_3^{-1}) s_4^{-1} w_0 s_4 \\ &= u_2 u_4 s_3 s_2^{-1} u_1 s_2^{-1} s_3 s_2 s_4^{-1} w_0 s_4 = u_2 u_4 s_3 s_2^{-1} u_1 s_2^{-1} s_3 s_4^{-1} w_0 s_4 s_2, \end{aligned}$$

so we are reduced to studying

$$\begin{aligned} u_4 s_3 (s_2^{-1} u_1 s_2^{-1}) s_3 s_4^{-1} w_0 s_4 &\subset u_4 s_3 u_1 s_2 s_1^{-1} s_2 s_3 s_4^{-1} w_0 s_4 + u_4 s_3 u_1 u_2 u_1 s_4^{-1} w_0 s_4 \\ &\subset u_1 u_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} w_0 s_4 + u_1 u_4 s_3 u_2 u_1 s_4^{-1} w_0 s_4 \\ &\subset V_0 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^+ s_4^{-1} w^+ s_4 A_4. \end{aligned}$$

Since $s_4^{-1} w^+ s_4^{-1} w^+ s_4 \in V_0$ by lemma 6.11 (8) (after applying Φ), this concludes the proof of (2). \square

Lemma 7.11. $s_4(s_3 s_2^{-1} s_3) s_1 s_2 s_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 \in V^+$

We have $s_4(s_3 s_2^{-1} s_3) (s_1 s_2 s_1) s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 = s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_2 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 = s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 (s_2 s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4$ which belongs to $s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 s_3^{-1} s_2 s_3^{-1} s_2 s_4^{-1} w_0 s_4 + s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 + s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4$ by lemma 3.6. Now

$$s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 \subset V^+$$

by lemma 7.9 , while $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_3u_2s_4^{-1}w_0s_4 \subset V^+$ by lemma 7.10 (1). We are thus reduced to considering

$$s_4(s_3s_2^{-1}s_3)s_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 \in s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 \\ + s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4$$

by lemma 2.3. We have

$$s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = s_2s_4(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2 \in V^+$$

by lemma 7.8. We have $s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = u_2s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2$ and $s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ is a linear combination of the $s_4s_3^\alpha u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ for $\alpha \in \{0, 1, -1\}$. When $\alpha = 0$ we get $s_4u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = u_2s_1s_4^2s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by proposition 6.3 ; for $\alpha = 1$ we have

$$\begin{aligned} s_4s_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 &= (s_4s_3s_4)u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ &= s_3s_4s_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = s_3s_4(s_3u_2s_3^{-1})s_1s_2s_3^{-1}s_4^{-1}w_0s_4 \\ &= s_3s_4s_2^{-1}u_3s_2s_1s_2s_3^{-1}s_4^{-1}w_0s_4 = s_3s_2^{-1}s_4u_3(s_2s_1s_2)s_3^{-1}s_4^{-1}w_0s_4 \\ &= s_3s_2^{-1}s_4u_3s_1s_2s_1s_3^{-1}s_4^{-1}w_0s_4 = s_3s_2^{-1}s_1s_4u_3s_2s_3^{-1}s_4^{-1}w_0s_4s_1 \subset V^+ \end{aligned}$$

by proposition 6.3 ; for $\alpha = -1$ we have

$$\begin{aligned} s_4s_3^{-1}u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 &= (s_4s_3^{-1}s_4)u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ &\subset u_3^\times s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ &\quad + u_3u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \end{aligned}$$

by lemma 2.4, and $u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by lemma 7.10 (2). Finally,

$$\begin{aligned} s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 &= s_4^{-1}s_3u_2s_1s_4^{-1}s_3^{-1}s_2(s_3^{-1}s_4^{-1}s_3)s_2s_1^2s_2s_3s_4 \\ &= s_4^{-1}s_3u_2s_1s_4^{-1}s_3^{-1}s_2s_4s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4 = s_4^{-1}s_3u_2s_1(s_4^{-1}s_3^{-1}s_4)s_2s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) \\ &= s_4^{-1}s_3u_2s_1s_3s_4^{-1}s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} = s_4^{-1}s_3u_2s_3s_4^{-1}s_1s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \subset V^+ \end{aligned}$$

by lemma 7.4 (4).

7.6. Reduction to $s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4$. We apply the following relations of B_4 :

$$\begin{cases} s_3(s_2s_1^{-1}s_2)s_3^{-1} &= s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2 \\ s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3 &= s_2s_1(s_3^{-1}s_2s_3^{-1})s_1^{-1}s_2^{-1} \end{cases}$$

This yields

$$\begin{aligned} &s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}w_0s_4 \\ &= s_4s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2s_4s_2s_1(s_3^{-1}s_2s_3^{-1})s_1^{-1}s_2^{-1}s_4^{-1}w_0s_4 \\ &= s_2^{-1}s_1^{-1}s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4s_1^{-1}s_2^{-1}. \end{aligned}$$

Expanding s_2^2 we get

$$s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in R^\times s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ + Rs_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ + Rs_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4$$

We have $s_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$ by lemma 7.8, and

$$s_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$$

by lemma 7.11.

Lemma 7.12.

- (1) $s_4s_3s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
- (2) $s_4s_3s_2^{-1}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V_0$

Proof. We prove (1). Using braid relations we get

$$\begin{aligned} s_4 s_3 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 &= s_1^{-1} s_4 (s_3 s_2 s_3^{-1}) s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_1^{-1} s_4 s_2^{-1} s_3 s_2 s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 = s_1^{-1} s_2^{-1} (s_4 s_3 s_4^{-1}) s_2^2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_1^{-1} s_2^{-1} s_3^{-1} s_4 s_3 s_2^2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \end{aligned}$$

and this latter term $s_1^{-1} s_2^{-1} s_3^{-1} s_4 s_3 s_2^2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4$ belongs to V^+ by lemma 7.4 (4). We now prove (2). By using braid relations we get

$$\begin{aligned} s_4 s_3 (s_2^{-1} s_1^{-1} s_2) s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 &= s_4 s_3 s_1 s_2^{-1} s_1^{-1} s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_1 s_4 (s_3 s_2^{-1} s_3^{-1}) s_4^{-1} s_1^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 = s_1 s_4 s_2^{-1} s_3^{-1} s_2 s_4^{-1} s_1^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_1 s_2^{-1} (s_4 s_3^{-1} s_4^{-1}) s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 = s_1 s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_1 s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 (s_1^{-1} s_2 s_1) s_3 s_4^{-1} s_2 s_3 w_0 s_4 = s_1 s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 s_2 s_1 s_2^{-1} s_3 s_4^{-1} s_2 s_3 w_0 s_4 \\ &= s_1 s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 s_2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3 w_0 s_4 = s_1 s_2^{-1} s_3^{-1} s_4^{-1} s_3 s_2 s_2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3^2 s_2 s_1^2 s_2 s_3 s_4 \end{aligned}$$

and $s_4^{-1} s_3 s_2^2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3^2 s_1^2 s_2 s_3 s_4$ is a linear combination of $s_4^{-1} s_3 s_2^2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3^\alpha s_2 s_1^2 s_2 s_3 s_4$ for $\alpha \in \{-1, 1, 0\}$. When $\alpha = 1$, we get $s_4^{-1} s_3 s_2^2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} s_3 s_2 s_1^2 s_2 s_3 s_4 = s_4^{-1} s_3 s_2^2 s_1 s_2^{-1} s_3 s_2 s_4^{-1} w_0 s_4 = s_4^{-1} s_3 s_2^2 s_1 s_2^{-1} s_3 s_4^{-1} w_0 s_4 s_2$ which clearly (by expanding s_2^2) belongs to $V_0 + A_4 s_4^{-1} w^+ s_4^{-1} w_0 s_4 A_4 = V_0 + A_4 s_4^{-1} w^+ s_4^{-1} w^+ s_4 A_4 \subset V^+$ by lemma 6.11 (8) (take the image by Φ of the identity there). When $\alpha \in \{0, -1\}$ we write $s_4^{-1} s_3 s_2^2 s_1 (s_2^{-1} s_3 s_2) s_4^{-1} s_3^\alpha s_2 s_1^2 s_2 s_3 s_4 = s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_3^{-1} s_4^{-1} s_3^\alpha s_2 s_1^2 s_2 s_3 s_4$. When $\alpha = 0$, we get $s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_3^{-1} s_4^{-1} (s_2 s_1^2 s_2) s_3 s_4 = s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_3^{-1} s_2 s_1^2 s_2 (s_4^{-1} s_3 s_4) = s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} \in V_0$, so we can assume $\alpha = -1$. Then

$$\begin{aligned} s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 (s_3^{-1} s_4^{-1} s_3^{-1}) s_2 s_1^2 s_2 s_3 s_4 &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_4^{-1} s_2 s_1^2 s_2 s_3 s_4 \\ &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_2 s_1^2 s_2 (s_4^{-1} s_3 s_4) \\ &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_2 s_1^2 s_2 s_3 s_4 s_3^{-1} \end{aligned}$$

is a linear combination of $s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_2 s_1^\beta s_2 s_3 s_4 s_3^{-1}$ for $\beta \in \{-1, 0, 1\}$. When $\beta = 0$ we get $s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_2^2 s_3 s_4 s_3^{-1} \in V_0$ by lemma 7.4 (4) (taking the image by Ψ of the second identity there). When $\beta = 1$ we get

$$\begin{aligned} s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} (s_2 s_1 s_2) s_3 s_4 s_3^{-1} &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} s_1 s_2 s_1 s_3 s_4 s_3^{-1} \\ &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_1 s_4^{-1} s_3^{-1} s_2 s_3 s_4 s_3^{-1} s_1 \in V_0 \end{aligned}$$

by the same argument. When $\beta = -1$ we get

$$\begin{aligned} s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4 s_3^{-1} &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 s_4^{-1} (s_2 s_1) (s_3 s_2^{-1} s_3) (s_1^{-1} s_2^{-1}) s_4 s_3^{-1} \\ &= s_4^{-1} s_3 s_2^2 s_1 s_3 s_2 (s_2 s_1) s_4^{-1} (s_3 s_2^{-1} s_3) s_4 (s_1^{-1} s_2^{-1}) s_3^{-1} \in V_0 \end{aligned}$$

again by the same argument, and this concludes the proof. \square

7.7. Reduction to $s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4$. We have

$$\begin{aligned} s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_4 s_3^{-1} s_2 (s_3^{-1} s_4^{-1} s_3^{-1}) s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_4 s_3^{-1} s_2 s_4^{-1} s_3^{-1} s_4^{-1} s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 (s_4 s_3^{-1} s_4^{-1}) s_2 s_3^{-1} s_4^{-1} s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_3^{-1} s_4^{-1} (s_3 s_2 s_3^{-1}) s_4^{-1} s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_3^{-1} s_4^{-1} s_2^{-1} s_3 s_2 s_4^{-1} s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} s_1 (s_3 s_2^{-1} s_3^{-1}) s_1 s_4^{-1} s_2^{-1} s_3 s_4^{-1} s_2 s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_2 s_1 s_4^{-1} s_2^{-1} s_3 s_4^{-1} s_2 s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_1^{-1} s_2 s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} (s_1 s_2^{-1} s_1^{-1}) s_3^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} s_2^{-1} s_1^{-1} s_2 s_3^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= s_4 s_3 s_2^{-1} s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \end{aligned}$$

and, expanding s_2^{-2} , we get

$$\begin{aligned} s_4 s_3 s_2^{-2} s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 &\in R^\times s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &\quad + R s_4 s_3 s_2^{-1} s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &\quad + R s_4 s_3 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \end{aligned}$$

and the last two terms belong to V_0 by lemma 7.12 (1) and (2).

Lemma 7.13.

- (1) $s_4(s_3 s_2^{-1} s_3) s_1 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \in V^+$
- (2) $s_4(s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \in V^+$
- (3) $s_4 s_3 s_2^{-1} s_3 s_1 s_2 s_4^{-1} s_3 s_4^{-1} \in A_4^\times s_4 s_3 s_2^{-1} s_3 s_4^{-2} A_3^\times$
- (4) $s_4 u_2 u_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \in V_0$

Proof. We prove (1). $s_4 s_3 s_2^{-1} s_3 s_1 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 = s_4 s_3 s_2^{-1} s_1 (s_3 s_4^{-1} s_3 s_4^{-1}) s_1 s_2 s_3 w_0 s_4$ belongs to

$$s_4 s_3 s_2^{-1} s_1 s_4^{-1} s_3 s_4^{-1} s_3 s_1 s_2 s_3 w_0 s_4 + s_4 s_3 s_2^{-1} s_1 u_3 u_4 s_1 s_2 s_3 w_0 s_4 + s_4 s_3 s_2^{-1} s_1 u_4 u_3 s_1 s_2 s_3 w_0 s_4$$

by lemma 3.6. We have $s_4 s_3 s_2^{-1} s_1 u_3 u_4 s_1 s_2 s_3 w_0 s_4 = s_4 s_3 s_2^{-1} u_3 u_4 s_1^2 s_2 s_3 w_0 s_4 \subset V^+$ by lemma 7.4

(4), and $s_4 s_3 s_2^{-1} s_1 u_4 u_3 s_1 s_2 s_3 w_0 s_4 = s_4 s_3 u_4 s_2^{-1} s_1 u_3 s_1 s_2 s_3 w_0 s_4 \subset V_0$ by lemma 6.11 (2). Moreover

$s_4 s_3 s_2^{-1} s_1 s_4^{-1} s_3 s_4^{-1} s_3 s_1 s_2 s_3 w_0 s_4 = (s_4 s_3 s_4^{-1}) s_2^{-1} s_1 s_3 s_4^{-1} s_3 s_1 s_2 s_3 w_0 s_4 = s_3^{-1} s_4 s_3 s_2^{-1} s_3 s_4^{-1} s_3 s_1^2 s_2 s_3 w_0 s_4 = s_3^{-1} s_4 s_3 s_2^{-1} s_3 s_4^{-1} s_3 s_1^2 s_2 s_3 w_0 s_4 \in V^+$ by lemma 7.4 (4). This proves (1). Using only braid relations we get

$$\begin{aligned} &s_4 s_3 s_2^{-1} s_3 s_1 s_2 s_4^{-1} s_3 s_4^{-1} &= s_1 s_1^{-1} s_4 s_3 s_2^{-1} s_1 s_3 s_2 s_4^{-1} s_3 s_4^{-1} \\ &= s_1 s_4 s_3 (s_1^{-1} s_2^{-1} s_1) s_3 s_2 s_4^{-1} s_3 s_4^{-1} &= s_1 s_4 s_3 s_2 s_1^{-1} (s_2^{-1} s_3 s_2) s_4^{-1} s_3 s_4^{-1} \\ &= s_1 s_4 s_3 s_2 s_1^{-1} s_3 s_2 s_3^{-1} s_4^{-1} s_3 s_4^{-1} &= s_1 s_4 (s_3 s_2 s_3) s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_3 s_4^{-1} \\ &= s_1 s_4 s_2 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_3 s_4^{-1} &= s_1 s_2 s_4 s_3 s_2 s_1^{-1} s_2 (s_3^{-1} s_4^{-1} s_3) s_4^{-1} \\ &= s_1 s_2 s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4^{-1} s_4^{-1} &= s_1 s_2 (s_4 s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4^{-2} \\ &= s_1 s_2 s_3 s_4 (s_3 (s_2 s_1^{-1} s_2) s_3^{-1}) s_4^{-2} &= s_1 s_2 s_3 s_4 (s_2^{-1} s_1^{-1}) (s_3 s_2^{-1} s_3) (s_1 s_2) s_4^{-2} \\ &= (s_1 s_2 s_3 s_2^{-1} s_1^{-1}) s_4 s_3 s_2^{-1} s_3 s_4^{-2} (s_1 s_2) \end{aligned}$$

which proves (3). From this we deduce that $s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \in A_4 s_4 u_3 u_2 u_3 u_4 A_4 s_4 \subset V^+$ by lemma 7.4 (4). This proves (2). Finally

$$\begin{aligned} s_4 u_2 u_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 &= u_2 s_1^{-1} s_4 u_3 s_4^{-1} s_2 s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &\subset u_2 s_1^{-1} s_3^{-1} u_4 (s_3 s_2 s_3) s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &\subset u_2 s_1^{-1} s_3^{-1} u_4 s_2 s_3 s_2 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &\subset u_2 s_1^{-1} s_3^{-1} s_2 u_4 s_3 s_4^{-1} s_2 s_1^2 s_2 s_3 w_0 s_4 \subset V_0 \end{aligned}$$

by lemma 6.11 (2). This proves (4). □

Lemma 7.14.

- (1) $u_4 A_4 s_4^{-1} s_3^\beta w_0 s_4 \subset V^+$
- (2) $s_4 u_3 u_2 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \subset V^+$

Proof. We prove (1). Using $A_4 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 + A_3 w^+ + A_3 w^-$ we get that $u_4 A_4 s_4^{-1} s_3^\beta w_0 s_4$ is the sum of the following abelian groups :

- $u_4 A_3 u_3 A_3 s_4^{-1} s_3^\beta w_0 s_4 = A_3 u_4 u_3 s_4^{-1} A_3 s_3^\beta w_0 s_4 \subset V_0$ by lemma 6.11 (2).
- $u_4 A_3 w^+ s_4^{-1} s_3^\beta w_0 s_4 = A_3 u_4 w^+ s_4^{-1} s_3^\beta w_0 s_4$, which is included in $V_0 + A_4 u_4 w^+ s_4^{-1} w^+ s_4 A_4$ by lemma 6.8 (4) and proposition 6.3. Now $u_4 w^+ s_4^{-1} w^+ s_4 \subset V_0 + R s_4 w^+ s_4^{-1} w^+ s_4 + R s_4^{-1} w^+ s_4^{-1} w^+ s_4$ and we have $s_4^{-1} w^+ s_4^{-1} w^+ s_4 \in V_0$ by lemma 6.11 (7) (apply $\Phi \circ \Psi$ to the identity there), so $u_4 A_3 w^+ s_4^{-1} s_3^\beta w_0 s_4 \subset V^+$
- $u_4 A_3 w^- s_4^{-1} s_3^\beta w_0 s_4 = A_3 u_4 w^- s_4^{-1} s_3^\beta w_0 s_4$, which is included in $V_0 + A_4 u_4 w^- s_4^{-1} w^+ s_4 A_4$ by lemma 6.8 (4) and proposition 6.3. Since $u_4 w^- s_4^{-1} w^+ s_4 \subset V_0$ by lemma 6.11 (5) we get $u_4 A_3 w^- s_4^{-1} s_3^\beta w_0 s_4 \subset V^+$
- $u_4 A_3 u_3 u_2 u_3 A_3 s_4^{-1} s_3^\beta w_0 s_4 = A_3 u_4 u_3 u_2 u_3 s_4^{-1} A_3 s_3^\beta w_0 s_4 \subset V^+$ by lemma 7.4 (4).

This proves (1). Since $s_4 u_3 u_2 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 = s_4 u_3 s_4^{-1} u_2 s_1^{-1} s_2 s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 = s_3^{-1} u_4 s_3 u_2 s_1^{-1} s_2 s_3 s_1^2 s_2 s_4^{-1} s_3 w_0 s_4 \subset s_3^{-1} u_4 A_4 s_4^{-1} s_3 w_0 s_4 \subset V^+$ by (1), and this proves (2). \square

7.8. Reduction to $s_4 w^- s_4 w_0^2 s_4$. We have

$$\begin{aligned} s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 &= s_4 (s_2^{-1} s_1^{-1}) (s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &= (s_2^{-1} s_1^{-1}) s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \end{aligned}$$

and, expanding s_2^2 , we get that this last element belongs to

$$\begin{aligned} &R^\times (s_2^{-1} s_1^{-1}) s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &+ R(s_2^{-1} s_1^{-1}) s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \\ &+ R(s_2^{-1} s_1^{-1}) s_4 (s_3 s_2^{-1} s_3) s_1 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4. \end{aligned}$$

Now $s_4 (s_3 s_2^{-1} s_3) s_1 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \in V^+$ by lemma 7.13 (1) and

$$s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 \in V^+$$

by lemma 7.13 (2). We are thus reduced to

$$\begin{aligned} s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_4^{-1} s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4 &= s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_4^{-1} s_3 s_4^{-1} s_1^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3) (s_1 s_2^{-1} s_1^{-1}) s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_4 (s_3 s_2^{-1} s_3 s_2^{-1}) s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \end{aligned}$$

which, by lemma 3.6, lies in

$$\begin{aligned} &R^\times s_4 s_2^{-1} s_3 s_2^{-1} s_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &+ s_4 u_2 u_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &+ s_4 u_3 u_2 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4. \end{aligned}$$

The two latter terms lie in V^+ by lemma 7.13 (4) and 7.14 (2), so we are reduced to

$$\begin{aligned} s_4 s_2^{-1} s_3 s_2^{-1} s_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 &= s_2^{-1} s_4 s_3 s_2^{-1} s_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} (s_3 s_4 s_3) s_2^{-1} s_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_4 s_3 s_4 s_2^{-1} s_3 s_1^{-1} s_2 s_4^{-1} s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_4 s_3 s_2^{-1} (s_4 s_3 s_4^{-1}) s_1^{-1} s_2 s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_4 (s_3 s_2^{-1} s_3^{-1}) s_4 s_3 s_1^{-1} s_2 s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_4 s_2^{-1} s_3^{-1} s_2 s_4 s_3 s_1^{-1} s_2 s_3 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_2^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_4 (s_3 s_2 s_3) s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_2^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_4 s_2 s_3 s_2 s_4^{-1} s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_2^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 (s_4 s_3 s_4^{-1}) s_2 s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_2^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_3 s_2 s_1^2 s_2 s_3 w_0 s_4 \\ &= s_2^{-1} s_3^{-1} s_2^{-1} s_4 w^- s_4 w_0^2 s_4 \end{aligned}$$

hence to $s_4 w^- s_4 w_0^2 s_4$.

7.9. Conclusion of the computation. By lemma 4.9, we have $w_0^2 \in A_3^\times w_0^{-1} + U^+ = w_0^{-1} A_3^\times + U^+$, and $U^+ = A_3 w_0 + U_0 = w_0 A_3 + U_0$. We then have $s_4 w^- s_4 w_0^2 s_4 \in s_4 w^- s_4 w_0^{-1} s_4 A_3^\times + s_4 w^- s_4 w_0 s_4 A_3 + s_4 w^- s_4 U_0 s_4$.

On the one hand, we know that $s_4 w^- s_4 A_3 u_3 A_3 s_4 = s_4 w^- s_4 A_3 u_3 s_4 A_3 = s_4 w^- s_4 u_2 u_1 u_2 u_1 u_3 s_4 A_3 = s_4 w^- s_4 u_2 u_1 u_2 u_3 s_4 u_1 A_3 \subset V_0$ by proposition 6.3, and that

$$\begin{aligned} s_4 w^- s_4 A_3 u_3 u_2 u_3 A_3 s_4 &= s_4 w^- s_4 A_3 u_3 u_2 u_3 s_4 A_3 \\ &= s_4 w^- s_4 u_1 u_2 u_1 (u_2 u_3 u_2 u_3) s_4 A_3 \\ &= s_4 w^- s_4 u_1 u_2 u_1 u_3 u_2 u_3 u_2 s_4 A_3 \\ &= s_4 w^- s_4 u_1 u_2 u_1 u_3 u_2 u_3 s_4 u_2 A_3 \subset V^+ \end{aligned}$$

by lemma 7.4 (4) (apply $\Phi \circ \Psi$ to the identity there). From $U_0 = A_3 u_3 A_3 + A_3 u_3 u_2 u_3 A_3$ one thus gets $s_4 w^- s_4 U_0 s_4 \subset V^+$.

On the other hand, we have $s_4 w^- s_4 w_0 s_4 \in V_0 + s_4 w^- s_4 w^+ s_4 A_3 \subset V_0$ by lemma 6.11 (5). We are thus reduced to $s_4 w^- s_4 w_0^{-1} \in s_4 w^- s_4 w^- s_4 A_3^\times + V_0$, which concludes the proof.

CONTENTS

1. Introduction	1
1.1. Perspectives	2
1.2. Applications	3
2. Preliminaries and notations	4
3. The algebra A_3	5
4. The algebra A_4 as a A_3 (bi)module	6
5. The algebra A_4 as a $\langle s_1, s_3 \rangle$ (bi)module	10
6. The algebra A_5	16
6.1. The A_4 -bimodule $A_5^{(3)}/A_5^{(2)}$: first reduction.	17
6.2. The A_4 -bimodule $A_5^{(3)}/A_5^{(2)}$: a smaller set of generators.	23
6.3. Image of the center of the braid group in $A_5^{(3)}/A_5^{(2)}$	29
6.4. Right actions are left actions	30
6.5. A_5 as a A_4 -module	31
7. Proof of lemma 6.16	33
7.1. Reduction to $s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4$	33
7.2. Reduction to $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^2 (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$	34
7.3. Reduction to $s_4 s_3 (s_2 s_1^2 s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$	36
7.4. Reduction to $s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2 s_1^2 s_2) s_3 s_4^{-1} w_0 s_4$	37
7.5. Reduction to $s_4 s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} w_0 s_4$	38
7.6. Reduction to $s_4 (s_3 s_2^{-1} s_3) s_1 s_2^{-1} s_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4$	40
7.7. Reduction to $s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4^{-1} s_2 s_3 s_4^{-1} s_1 s_2 s_3 w_0 s_4$	41
7.8. Reduction to $s_4 w^- s_4 w_0^2 s_4$	43
7.9. Conclusion of the computation	43
References	44

REFERENCES

- [1] Assion, J., Einige endliche Faktorgruppen der Zopfgruppen, *Math. Z.*, **163**, (1978), 291–302.
- [2] P. Bellingeri, L. Funar, *Polynomial invariants of links satisfying cubic skein relations*, *Asian J. Math.* **8** (2004), 475–509.
- [3] M. Broué, G. Malle, *Zyclotomische Heckealgebren*, in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, *Astérisque* **212** (1993), 119–189.
- [4] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, *J. Reine Angew. Math.* **500** (1998), 127–190.
- [5] M. Cabanes, I. Marin, *On ternary quotients of cubic Hecke algebras*, preprint 2010.
- [6] W.L. Chow, *On the algebraic braid group*, *Ann. of Math.* **49** (1948), 654–658.
- [7] H.S.M. Coxeter, *Factor groups of the braid groups*, *Proc. Fourth Canad. Math. Congress*, (1957), 95–122.
- [8] P. Etingof, E. Rains, *Central extensions of preprojective algebras, the quantum Heisenberg algebra, and 2-dimensional complex reflection groups*, *J. Algebra* **299** (2006), 570–588.
- [9] L. Funar, *On the quotients of cubic Hecke algebras*, *Comm. Math. Phys.* **173** (1995), 513–558.
- [10] G. Malle, *On the rationality and fake degrees of cyclotomic Hecke algebras*, *J. Math. Sci. Univ. Tokyo* **6** (1999), 647–677.
- [11] I. Marin, *Krammer representations for complex braid groups*, arXiv:0711.3096 v3 (2008).
- [12] J. Müller, *On exceptional cyclotomic Hecke algebras*, preprint 2004.
- [13] I. Tuba, H. Wenzl, *Representations of the braid group B_3 and of $SL_2(\mathbf{Z})$* , *Pacific J. Math.* **197** (2001), 491–510.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PARIS 7

E-mail address: marin@math.jussieu.fr